Real Analysis

- One of principal areas in mathematics.
- A branch of mathematical analysis which includes real analysis, complex analysis, functional analysis, differential equations, measure theory.
- Deals w/ real #'s and real-valued functions.
- Provides theoretical justification of Calculus: limits, continuity, series, differentiation, integration

• Historical background of mathematical analysis

- Greeks! implicitly used analysis ideas:
  e.g., in the method of exhaustion to compute the area of a shape using a sequence of polygons (Archimedes)

- India, 14th cent.: infinite series expansions of some trig functions
- Europe, 17th cent: modern analysis began!
  Descartes, Fermat, Newton, Leibnitz
  (analytical geom.) (calculus)

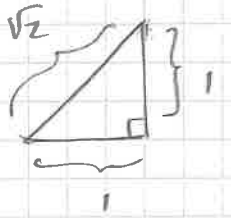
- 18th, 19th cent.
  - Euler: function
  - Bolzano: continuity
  - Cauchy: Cauchy sequence, complex analysis
  - Poisson, Fourier: PDE, harmonic analysis
  - Riemann: integration
  - Weierstrass: "father of modern analysis" (ε, δ) def. of limit, important theorems, calculus of variations

- Later: Cantor, Dedekind, Baire, Lebesgue, Hilbert, Banach
  ↓
  functional analysis ...

Abbott "Understanding Analysis"

Chapter 1 The Real Numbers

§1.1 Irrationality of $\sqrt{2}$.



Greeks: discovered that the diagonal of a unit square is not a rational number.

Greeks (implicitly) interpreted numbers to mean rational numbers, so this discovery forced them to think that "arithmetic number" is a weaker concept than "length" (they assumed that the length of one line segment is a rational multiple of the length of another line segment).

Modern approach: strengthen the concept of "number". Start with

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ natural numbers: addition only.

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ integers: additive identity (0) and additive inverses $\Rightarrow$ subtraction.

$\mathbb{Q} = \{ p/q \mid p, q \in \mathbb{Z},\ q \neq 0 \}$ rational numbers: additive identity, inverses; multiplication; multiplicative identity (1) & multiplicative inverses

$\mathbb{Q}$ is a field (for comparison, $\mathbb{N}$ & $\mathbb{Z}$ are not fields)

↓

a set with $+$, $\cdot$ and (with well-defined properties such as commutativity, associativity, distributivity)

inverses

⇓

$-$, $/$

$\mathbb{Q}$: given $r, s \in \mathbb{Q}$, we have $r<s$, $r=s$, or $r>s$

⇒ $\mathbb{Q}$ has a natural order defined on it.

If $r<s$, $s<t$ ⇒ $r<t$ ⇒ rational #'s lay out on the number line from left to right.

Rational #'s are dense: if $r<s$, then $\frac{r+s}{2}$ is halfway between r & s.

$\mathbb{Q}$ is pretty good, but what about $\sqrt{2}$?

As we said earlier, $\sqrt{2} \notin \mathbb{Q}$. Let's prove it:

Theorem: There is no rational number whose square is 2.

Proof: (by contradiction)

Suppose there is a rational number whose square is 2, that is, there exist integers p & q satisfying $\left(\frac{p}{q}\right)^2 = 2$. We assume that p & q have no common factors (if there are any common factors, we could simply cancel them). So, $\left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} = 2$

⇒ $p^2 = 2q^2$. This implies that p^2 is even, and hence p is also even (see this proof in the handout "On proofs").

Thus, $p = 2k$, $k \in \mathbb{Z}$, and $p^2 = 4k^2 = 2q^2$ yields $2k^2 = q^2$. The last equation implies that q^2, and therefore, q are even. Thus, we have shown that both p and q are even which contradicts our assumption that they have no common factors! So, $\left(\frac{p}{q}\right)^2 = 2$ cannot hold for any integers p & q. □

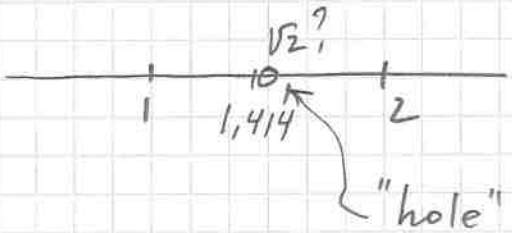
Thus, $\sqrt{2} \notin \mathbb{Q}$.

We can approximate $\sqrt{2}$ w/ rational numbers:

$1,414^2 = 1.999396$

$1,41414^2 = 1,99979194$

...



⇒ We need extension to our chain

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}$

by appending real numbers:

$\mathbb{R}$

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

↓

?

How to construct $\mathbb{R}$ from $\mathbb{Q}$?

What are numbers from $\mathbb{R}$ that are irrational?

Is there a way to express irrational numbers?

Is there any symmetry between the rationals and irrationals?