

## § 1.2 Preliminaries

(1)

### Sets

- A set is a collection of objects.

Ex:

$$S = \{ \circ, \Delta, \square, \odot \}, \quad S = \{ 1, 2, 3, 4, 5 \}, \quad S = \mathbb{Z}.$$

- We write  $x \in S$  if  $x$  is an element of  $S$ .

If  $x$  is not in  $S$ , we write  $x \notin S$ .

- The empty set:  $\emptyset$  (no elements)

- $A \subseteq B$  means "A is a subset of B", that is if  $x \in A$  then  $x \in B$  or  $x \in A \Rightarrow x \in B$ .

"implies"  $(A \subseteq B)$

- If  $A \subseteq B$  and  $A \neq B$  then  $A$  is a proper subset of  $B$ .

- If  $A \subseteq B$  and  $B \subseteq A$ , then  $A = B$   
(else  $A \neq B$ )

- Union of  $A$  &  $B$ :  $A \cup B = \{ x : x \in A \text{ or } x \in B \}$   
 $= \{ x \mid x \in A \text{ or } x \in B \}$   
or

- Intersection of  $A$  &  $B$ :  $A \cap B = \{ x : x \in A \text{ and } x \in B \}$

- Complement of  $A$  :  $A^c = \{ x : x \notin A \}$

- Subtraction :  $A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$   
"A minus B"

- Product of  $A$  &  $B$ :  $A \times B = \{ (a, b) : a \in A \text{ and } b \in B \}$   
 $\underbrace{\text{order matters!}}$

Examples:  $\mathbb{N} = \{ 1, 2, 3, 4, \dots \}$

and let  $E$  be the set of even natural #'s  
and let  $O$  be the set of odd natural #'s.

Then  $N \cup E = N$ ,  $N \cap E = E$ ,  $E \cup O = N$ ,  
 $E \cap O = \emptyset$  ( $E$  &  $O$  are disjoint sets). (2)

Also:  $E \subset N$ ,  $O \subset N$   
 $E \neq O$  and  $O \neq E$  (i.e.,  $E \neq O$ )

- More notations:

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

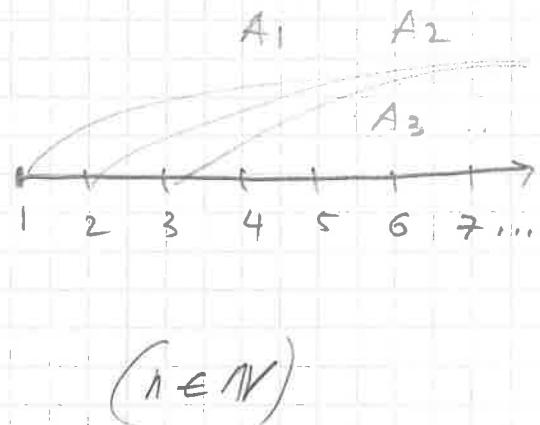
Example (textbook, 1.2.2)

$$A_1 = N = \{1, 2, 3, \dots\}$$

$$A_2 = \{2, 3, 4, \dots\}$$

$$A_3 = \{3, 4, 5, \dots\}$$

$$\dots A_n = \{n, n+1, \dots\}$$



$$(n \in N)$$

Then  $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n \in N} A_n = A_1 \cup A_2 \cup A_3 \cup \dots = A_1 = N$$

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n \in N} A_n = A_1 \cap A_2 \cap A_3 \cap \dots = \emptyset$$

(Why? Say, we have  $m \in \bigcap_{n=1}^{\infty} A_n$ , then  $m \in A_n \forall n \in N$ , but  $m \notin A_{m+1} = \{m+1, m+2, \dots\}$ )

- We will be working w/  $\mathbb{R}$ , and subsets of  $\mathbb{R}$ .  
 revisit:  $A^c = \{x \in \mathbb{R}: x \notin A\}$  universal set

- De Morgan's Laws :

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$
(HW1)

## Functions

Def: A function from a set  $A$  to a set  $B$  is a rule that assigns a single element of  $B$  to every element of  $A$ .

$$f: A \rightarrow B$$

for  $x \in A$ ,  $f(x) \in B$ .

The domain of  $f$ : set  $A$ ; the range of  $f$  is not necessarily  $B$ , it is  $\{y \in B : y = f(x), x \in A\}$

(The notion of "function": Euler, Fourier, Lobachevsky, Dirichlet  $\leftarrow$  modern definition.)

Example: (Dirichlet, 1829)

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

Domain:  $\mathbb{R}$   
Range:  $\{0, 1\}$

Example: Absolute value function

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain:  $\mathbb{R}$   
Range:  $[0, \infty)$

Properties of  $|x|$ :

$$(1) |ab| = |a||b| \leftarrow \forall a, b \in \mathbb{R}$$

$$(2) |a+b| \leq |a| + |b| \leftarrow \text{the triangle inequality}$$

$\hookrightarrow$  Proof: (using cases — by exhaustion)

First,  $\forall a \in \mathbb{R}$ ,  $a \leq |a|$ . Indeed, if  $a > 0$ , then  $a \leq |a| = a$ . If  $a \leq 0$ , then  $a \leq 0 \leq |a|$ .

Applying inequality  $a \leq |a|$  to  $-a$  yields  $-a \leq |-a| = |a|$  (we'll use this later).

Next, let  $a, b \in \mathbb{R}$ . If  $a+b > 0$ , then

$$|a+b| = a+b \leq |a| + |b|. \quad (a \leq |a|, b \leq |b|)$$

On the other hand, if  $a+b \leq 0$ , then

$$|a+b| = -(a+b) = -a - b \leq |a| + |b| \text{ again.}$$

This completes the proof.  $\square$

Note: for  $a, b, c \in \mathbb{R}$ ,  $|a-b| = |(a-c)+(c-b)|$

$$\Rightarrow |(a-c)+(c-b)| \leq |a-c| + |c-b| \Rightarrow$$
$$|a-b| \leq \underbrace{|a-c|}_{\substack{\text{distance} \\ \text{from } a \text{ to } b}} + \underbrace{|c-b|}_{\substack{\text{distance} \\ \text{from } a \text{ to } c}} \quad (\text{by triangle inequality})$$

Another proof exercise:

Theorem: Two real numbers  $a$  and  $b$  are equal if and only if  $\forall \epsilon > 0 \quad |a-b| < \epsilon$ .

Proof: "if and only if" ( $\Leftrightarrow$ ) means we need to prove 2 statements:

$(\Rightarrow)$  "If  $a=b$ , then  $\forall \epsilon > 0 \quad |a-b| < \epsilon$ ."

If  $a=b \Rightarrow |a-b|=0 \Rightarrow \forall \epsilon > 0 \quad |a-b|=0 < \epsilon$ .

$(\Leftarrow)$  The converse statement:

"If  $\forall \epsilon > 0 \quad |a-b| < \epsilon \Rightarrow a=b$ ."

Proof by contradiction: we assume  $a \neq b$ ,

then  $\epsilon_0 = |a-b| > 0$  conflicts with the assumption that  $|a-b| < \epsilon \quad \forall \epsilon > 0$ , that is,

$|a-b| < \epsilon_0$  and  $|a-b| = \epsilon_0$  cannot both be true!

Thus,  $a=b$ , and the proof is complete.  $\square$

- Read on induction, p.p. 10-11