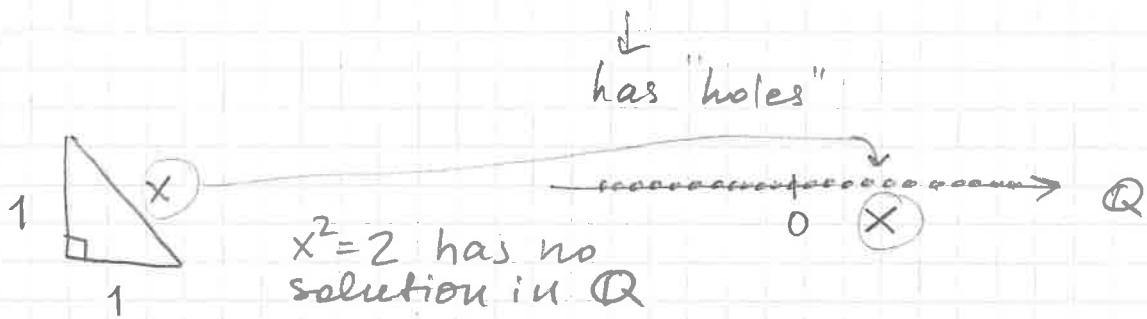


§1.3 The Axiom of Completeness. ①

Recall: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subseteq \mathbb{R}$



\mathbb{R} is an extension of \mathbb{Q} with no holes/gaps.
(not a "good" definition of \mathbb{R} , a rigorous construction of \mathbb{R} is in §8.6)

Now: $\mathbb{R} \supseteq \mathbb{Q}$, \mathbb{R} has operations of addition & multiplication; $\forall x \in \mathbb{R}$, x has an additive inverse and a multiplicative inverse.
 $(-x)$ $(\frac{1}{x})$
0 & 1 are additive and multiplicative identities respectively.

→ commutative, associative, distributive properties hold
In fact, \mathbb{R} is a field (as well as \mathbb{Q}).

\mathbb{R} is ordered: $\forall x, y \in \mathbb{R}$, $x < y$ or $x = y$ or $x > y$ and if $x < y$, $y < z \Rightarrow x < z$.

What about holes in \mathbb{Q} , that we don't have in \mathbb{R} ?

- Axiom of Completeness: Every nonempty set of real numbers that is bounded above has a least upper bound.

Need the following definitions:

Def: Least Upper Bound & Greatest Lower Bound.

A set $A \subset \mathbb{R}$ is bounded above if

$\exists b \in \mathbb{R}$ s.t. $a \leq b \forall a \in A$.
 "there exists" "such that" "forall"

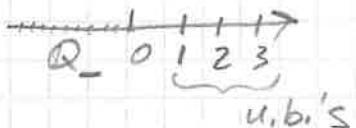
b is called an upper bound for A (u.b.)

Similarly, A is bounded below if $\exists l \in \mathbb{R}$ s.t. $l \leq a \forall a \in A$, and l is a lower bound (l.b.)

Example: Let $A = \mathbb{Q}_-$ = negative rationals,

Has upper bound 1 and many others.

Def: Let s be a real # s.t.:



① s is an upper bound for $A \subset \mathbb{R}$;

② if b is any upper bound for A , then

$$s \leq b$$

Then s is called the least upper bound of A

or supremum of A . Notation: or
$$\boxed{s = \text{lub } A}$$

$$\boxed{s = \sup A}$$

Define the greatest lower bound or
infimum ($\text{lub } A$) ($\text{glb } A$)

Examples:

$$\left\{ \begin{array}{l} \text{intuitively} \\ \sup \left\{ \frac{1}{2}, 1, 2 \right\} = 2 \\ \sup \mathbb{Q}_- = 0 \\ \sup \{ r \in \mathbb{R}, r < 2 \} = 2 \\ \sup \{ r \in \mathbb{R}, r^2 < 2 \} = \sqrt{2} \\ \sup [0, 1] = 1 \\ \sup (0, 1) = 1 \end{array} \right.$$

Example (text, 1.3.3)

(3)

$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$ is bounded above and below. \exists many upper bounds. We claim $\sup A = 1$. Why? First, $\forall n \in \mathbb{N}$, $\frac{1}{n} \leq 1$, so 1 is an upper bound. If b is any upper bound, then $b \geq 1$ since $1 \in A$. Thus, $\sup A = 1$.

It can be shown $\inf A = 0$ (using Thm. 1.4.2, Archimedean property)

Important: $\sup A$ and $\inf A$ may or may not be elements of A ! Compare to max/min:

Def: A real number M is a maximum of A if $M \in A$ and $a \leq M \quad \forall a \in A$. Similarly, $m \in \mathbb{R}$ is a minimum of A if $m \in A$ and $m \leq a \quad \forall a \in A$.

Example: $\sup (0, 1) = \sup [0, 1] = 1 = \max [0, 1]$
 $\max (0, 1)$ does not exist!

• Maximum is a specific type of upper bound that is required to belong to the set.

(Max is sup, but sup does not have to be max)

Note: sup and inf are unique. Why?
Say, $s_1 = \sup A$ & $s_2 = \sup A$. Then $s_1 \leq s_2$ and $s_2 \leq s_1$, by definition. Hence, $s_1 = s_2$. (Similar for inf).

Back to Axiom of Completeness:

(Every nonempty subset of real #'s that is bounded above, has a least upper bound.)

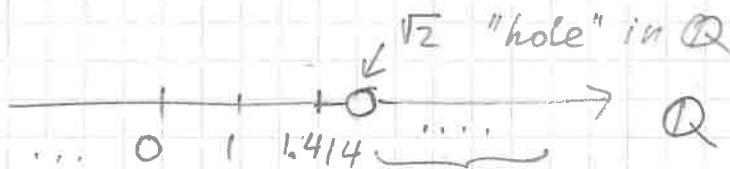
axiom = accepted assumption (so, no proof)

- \mathbb{R} has the least upper bound (LUB) property, but \mathbb{Q} does not:

$$A = \{r \in \mathbb{Q}, r^2 < 2\} \subset \mathbb{Q}$$

A is bounded above (many upper bounds, e.g., $2, \frac{3}{2}, 10, \dots$)

$\sup A$? Recall $\sqrt{2} \approx 1.4142\dots \in \mathbb{R}$ and we have a hole in \mathbb{Q} in place of $\sqrt{2}$ in \mathbb{R} . We may try many numbers close to $\sqrt{2}$, say $1.42, 1.415, 1.4143, \dots$, they will bound A above, but for each of them, there is always a smaller one! (Recall: \mathbb{Q} is dense)



many upper bounds for A
but $\nexists \sup A$

If $A = \{r \in \mathbb{R}, r^2 < 2\}$, then $\sup A = \sqrt{2}$.

In fact, we construct \mathbb{R} s.t. it has the LUB property; taking \mathbb{Q} and "filling in" the gaps.

Note: we will discuss more tools (in the next section) to be able to explain how \mathbb{Q} (and \mathbb{N}) fit inside \mathbb{R} .

Properties of sup: $A, B \subseteq \mathbb{R}$

(5)

① If $A \subseteq B$ then $\sup A \leq \sup B$.

Why? $A \subseteq B \Rightarrow \text{if } a \in A \Rightarrow a \in B \Rightarrow$

$$\left\{ \begin{array}{l} a \leq \sup B \quad (\text{by def of sup } B) \\ \text{That is, } \sup B \text{ is an upper bound for } A \end{array} \right.$$

$\left| \begin{array}{l} \text{By def. of sup } A, \sup A \leq \sup B \\ (1, 4, 6) \quad (\text{u.b.}) \end{array} \right. \square$

② To show $\sup A = \sup B$,

show " \leq " and " \geq " and the following: (def)
 d is u.b. for $A \Leftrightarrow \sup A \leq d$

③ If $A \subseteq \mathbb{R}$ is bounded above, $A \neq \emptyset$

and $c \in \mathbb{R}$, and $c+A = \{c+a : a \in A\}$ then
 $\sup(c+A) = c + \sup A$. (def.)

Why? $\left\{ \begin{array}{l} \text{let } s = \sup A, \text{ then } \forall a \in A, a \leq s \\ \Rightarrow c+a \leq c+s \quad \forall a \in A. \text{ Thus,} \end{array} \right.$

$\underline{\text{c+s is an u.b. for } c+A}$. Now,
 $\underline{\text{if } b \text{ is an u.b. for } c+A, \text{ then}}$

$$c+a \leq b \quad \forall a \in A \Rightarrow a \leq b-c \quad \forall a \in A$$

$\Rightarrow b-c$ is an u.b. for A . Since $s = \sup A$,

$$\text{then } s \leq b-c \Rightarrow c+s \leq b \text{ or}$$

$$\underline{c+\sup A \leq b} \Rightarrow \sup(c+A) = c + \sup A. \quad (\forall b \text{ u.b.}) \quad \square$$

④ If $A+B = \{a+b : a \in A, b \in B\} \Rightarrow$

$$\sup(A+B) = \sup A + \sup B \quad (\text{proof using } ③)$$

⑤ Lemma 1.3.8: Let $s \in \mathbb{R}$ be an u.b. for $A \subseteq \mathbb{R}$. Then, $s = \sup A \Leftrightarrow \forall \varepsilon > 0 \ \exists a \in A \text{ s.t. } s - \varepsilon < a$.

⑥

Proof: $\Rightarrow)$ Assume $s = \sup A$ and consider $s - \epsilon$, $\epsilon > 0$. Since $s - \epsilon < s \Rightarrow s - \epsilon$ is not an u.b. for A , and hence $\exists a \in A$ s.t. $s - \epsilon < a$ (otherwise, $s - \epsilon$ would be an u.b.)

$\Leftarrow)$ Assume s is an u.b. and $\forall \epsilon > 0 \exists a \in A$ s.t. $s - \epsilon < a$, i.e., any number less than s ($s - \epsilon$) is not an u.b.! We are done, because any $b < s$ cannot be an u.b. for A , and therefore, $s = \sup A$.

□

- For greatest lower bounds, analogous versions of the results discussed above can be stated & proved.