

§ 1.4 Consequences of Completeness.

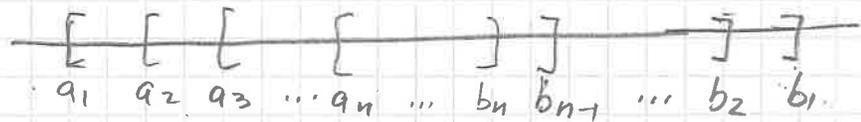
①

Theorem (1.4.1) Nested Interval Property

$\forall n \in \mathbb{N}$, assume we are given

$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that $I_n \supseteq I_{n+1}$. Then, the resulting nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ has a non-empty intersection; that is, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Proof:



Consider the set of left-hand endpoints of the intervals: $A = \{a_n : n \in \mathbb{N}\}$

Since $\dots \supseteq I_n \supseteq I_{n+1} \supseteq \dots$ is nested, any b_n serves as u.b. for A . Since A is bounded above, then by AOC (Axiom of Completeness), $\exists x = \sup A$.

For $I_n = [a_n, b_n]$ ($\forall n \in \mathbb{N}$), $a_n \leq x = \sup A$ and $x \leq b_n$ (u.b. of I_n). Thus, $\forall n \in \mathbb{N}$, $a_n \leq x \leq b_n \Rightarrow$

$x \in I_n \forall n \in \mathbb{N} \Rightarrow x \in \bigcap_{n=1}^{\infty} I_n \neq \emptyset$. □

- Thm. 1.4.1 is the 1st application of AOC. Now let us discuss how \mathbb{N} and \mathbb{Q} sit inside \mathbb{R} .

Theorem (1.4.2) Archimedean Property

(1) Given $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > x$

(2) Given $\forall y \in \mathbb{R}$, $y > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y$.

Proof: (1) states that \mathbb{N} is not bounded above.

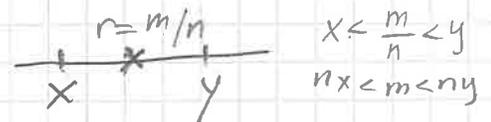
Suppose, for contradiction, that N is bounded above. (a)
 By the AOC, N should have a supremum, say, $d = \sup N$. By Lemma 1.3.8, $\forall \epsilon > 0 \exists n \in N$ s.t. $d - \epsilon < n$. Let $\epsilon = 1$ and \bar{n} be such a number: $d - 1 < \bar{n}$. Hence, $d < \bar{n} + 1 \in N$ which brings us to a contradiction: $\left(\begin{array}{l} \sup N < \bar{n} + 1 \in N. \\ \sup N \text{ is an u.b. for } N \end{array} \right.$

Thus, N is not bounded above. (Note: N is closed under addition.)

(2) Consider $\forall y > 0$ ($y \in \mathbb{R}$) and $\frac{1}{x} = \frac{1}{y}$. Then $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{x} = \frac{1}{y}$, that is, $\frac{1}{n} < y$. □

Theorem (1.4.3) Density of \mathbb{Q} in \mathbb{R} .

For any $x, y \in \mathbb{R}$ w/ $x < y$, $\exists r \in \mathbb{Q}$ s.t. $x < r < y$.



Proof: Let $x, y \in \mathbb{R}$ and $x < y$. Then $y - x > 0$.

By the Archimedean Property part (2), $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < y - x$. Now let us pick $m \in \mathbb{Z}$ s.t. $m - 1 \leq nx \leq m$ (i.e., m is the smallest integer $> nx$). From inequality (3), $x < \frac{m}{n}$. ☺

We can show that $\frac{m}{n} < y$:

① $\frac{1}{n} < y - x \Rightarrow x < y - \frac{1}{n}$

② $m - 1 \leq nx \Rightarrow m \leq nx + 1$

So, $m \leq nx + 1 < n(y - \frac{1}{n}) + 1 = ny - 1 + 1 = ny$

$\Rightarrow y > \frac{m}{n}$. ☺ & ☺ $\Rightarrow x < r < y$, where

$r = \frac{m}{n} \in \mathbb{Q}$. □

- The last result implies that \mathbb{Q} is dense in \mathbb{R} . (3)

Corollary (1.4.4) For any $x, y \in \mathbb{R}$, $x < y$, there exists an irrational number t s.t. $x < t < y$.

The Existence of Square Roots

- Recall: $A = \{r \in \mathbb{Q} : r^2 < 2\}$ has no supremum in \mathbb{Q} , but it does in \mathbb{R} .

Theorem (1.4.5) $\boxed{\exists d \in \mathbb{R}, d > 0 \text{ s.t. } d^2 = 2}$

Proof: Let $S = \{r \in \mathbb{R} : r^2 < 2\}$ and $d = \sup S$.

We will show that $d^2 = 2$ by ruling out the cases $d^2 < 2$ and $d^2 > 2$.

(1) Assume $d^2 < 2$ and consider $(\forall n \in \mathbb{N})$:

$$\begin{aligned} \left(d + \frac{1}{n}\right)^2 &= d^2 + 2d/n + 1/n^2 < d^2 + \frac{2d}{n} + \frac{1}{n} \\ &= d^2 + \frac{2d+1}{n} \end{aligned}$$

$\frac{1}{n^2} < \frac{1}{n}$

Now, choose $n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < \frac{2-d^2}{2d+1}$
 (Archimedean prop.) \Rightarrow

$$\frac{2d+1}{n_0} < 2-d^2 \Rightarrow$$

$$\Rightarrow \left(d + \frac{1}{n_0}\right)^2 < d^2 + \frac{2d}{n_0} + \frac{1}{n_0} = d^2 + \frac{2d+1}{n_0} < d^2 + 2-d^2 = 2$$

That is, $d + \frac{1}{n_0} \in S$ and $d + \frac{1}{n_0} > d$ (u.b.) \Rightarrow a contradiction. Thus, $d^2 < 2$ is not possible.

(2) Let $d^2 > 2$ and consider $\left(d - \frac{1}{n}\right)^2$
 $= d^2 - \frac{2d}{n} + \frac{1}{n^2} > d^2 - \frac{2d}{n}$

Now, let us pick n_0 w/ $\frac{1}{n_0} < \frac{d^2-2}{2d}$ \Rightarrow
 (Archim. prop.)

$$\frac{2d}{n_0} < d^2 - 2. \text{ Then } (d - \frac{1}{n_0})^2 > d^2 - \frac{2d}{n_0} >$$

(4)

$d^2 - (d^2 - 2) = 2$. Hence, $d - \frac{1}{n_0}$ is an u.b. for S . But clearly $d - \frac{1}{n_0} < d$ which contradicts the fact that $d = \sup S$. So, $d^2 > 2$ is not possible either. We are left w/ $d^2 = 2$
 $\Rightarrow d = \sqrt{2} \in \mathbb{R}$. □

Note: One can show that $\forall x \in \mathbb{R}, x > 0$,
 $\exists \sqrt{x} \in \mathbb{R}$ or even $\sqrt[m]{x} \in \mathbb{R}$ ($m \in \mathbb{N}$).