

## § 1.5 Cardinality.

$\mathbb{R}$ : a set of rational and irrational numbers continuously packed together along the number line. Both  $\mathbb{Q}$  and  $\mathbb{I}$  (set of irrationals) are dense in  $\mathbb{R}$  (§ 1.4), that is,  $\forall (a, b)$  in  $\mathbb{R}$  has both rational and irrational numbers. Are these mixed together in equal proportions? How do we compare sets?

$$\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$$

### 1-1 Correspondence

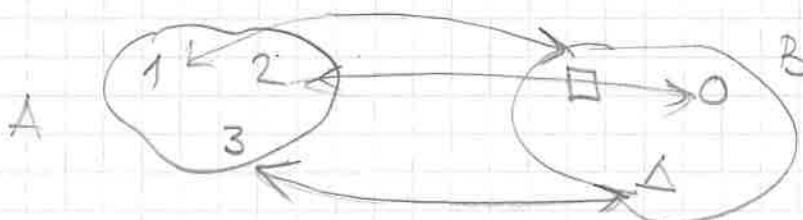
(Georg Cantor, 1845-1918)

Def: A function  $f: A \rightarrow B$  is one-to-one (1-1) if  $a_1 \neq a_2$  in  $A \Rightarrow f(a_1) \neq f(a_2)$  in  $B$ .  
 (Or: whenever  $f(a_1) = f(a_2) \Rightarrow \underbrace{a_1 = a_2}_{\text{in } B} \Rightarrow \underbrace{a_1 = a_2}_{\text{in } A}$ )

The function  $f$  is onto, if  $\forall b \in B \exists a \in A$  s.t.  $f(a) = b$ .

1-1 (injection)  
 onto (surjection)  $\Downarrow$  = bijection

A bijection puts  $A$  and  $B$  into "1-1 correspondence"



Each element of  $A$  is paired w/ exactly one element of  $B$ .

Recall: Cardinality is the size of a set (2) (card(A), |A|, n(A))

Def: Set A has the same cardinality as B if  $\exists f: A \rightarrow B$  that is 1-1 and onto.

We write  $A \sim B$ .  
(A is equivalent to B)

(A & B has a  
1-1 correspondence)

### Examples

1)  $\{\square, \circ, \Delta\} \sim \{1, 2, 3\}$

2)  $E = \underbrace{\{2, 4, 6, \dots\}}_{\text{even natural #'s}} \sim N$  Let  $f(n) = 2n$

(even if it's tempting to say  
 $E$  is "smaller" than  $N$   
since  $E \subset N$   
proper)

1	2	3	4	5	6	...
↓	↓	↓	↓	↓	↓	↓

2	4	6	8	10	12	...
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3)  $N \subset \mathbb{Z}$  Consider  $f: N \rightarrow \mathbb{Z}$  s.t.

(proper)  $f(1) = 0, f(2n) = n, f(2n+1) = -n$

$\Rightarrow f$  is a bijection (see another example in text)

(can you show this?), hence  $N \sim \mathbb{Z}$

$N$	1	2	3	4	5	6	7	...
	↑	↑	↑	↑	↑	↑	↑	

$\mathbb{Z}$	0	1	-1	2	-2	3	-3	...
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4)  $[0, 1] \sim [a, b]$  via  $x = g(t) = a + (b-a)t$   
 $t \mapsto x$  via  $g(0) = a, g(1) = b, \dots$

5)  $(-1, 1) \sim \mathbb{R}$  via  $f(x) = \frac{x}{x^2 - 1}$   
(or any  $(a, b)$ )

Def: A is finite if  $A \sim [1, 2, \dots, n]$  for some  $n \in \mathbb{N}$  (3)

or if A is empty.

Else, A is infinite.

## Countable Sets

Def: An infinite set A is countable, if  $A \sim \mathbb{N}$ , otherwise, A is uncountable.

Examples from above:  $\mathbb{E}, \mathbb{Z}$  are countable

( $\mathbb{N}$  is countable  $\therefore$ , use  $f(n) = n$ )

More:

A sequence  $\{x_1, x_2, x_3, \dots\}$  of distinct terms is countable:  $\downarrow \downarrow \downarrow$  Use  $f(n) = x_n$   
 $1 \quad 2 \quad 3$

So, any countable set can be listed in a (infinite) sequence.

|| Theorem: every infinite  $\overbrace{\text{countable}}$  subset of a countable set is countable. (See also Thm. 1.5.7)

Q: Are all infinite sets countable?  $\mathbb{Q}?$   $\mathbb{R}?$

(Can we put the elements of  $\mathbb{Q}$  and  $\mathbb{R}$  into a single list w/ 1-1 correspondence w/  $\mathbb{N}$ ?)

Theorem (1.5.6) (1)  $\mathbb{Q}$  is countable.

(2)  $\mathbb{R}$  is uncountable.

Proof: (1) Set  $A_1 = \{0\}$ , and for  $n \geq 2$ ,

$A_n = \left\{ \frac{p}{q} \mid p, q \in \mathbb{N} \text{ are in lowest terms, } w/ p+q=n \right\}$

Have:  $A_1 = \{0\}$ ,  $A_2 = \{\frac{1}{1}, -\frac{1}{1}\}$ ,  $A_3 = \{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}\}$ , ④  
 $A_4 = \{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}\}$ ,  $A_5 = \{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}\}$ ,  
so on.

$\forall n$ ,  $A_n$  is finite w/ any rational number appearing in exactly one of these sets!

$N$	1	2	3	4	5	6	7	8	9	10	11	...
(*) $\mathbb{Q}$	0	$\frac{1}{1}$	$-\frac{1}{1}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{2}{1}$	$-\frac{2}{1}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{3}{1}$	$-\frac{3}{1}$	...

$A_1$        $A_2$        $A_3$        $A_4$

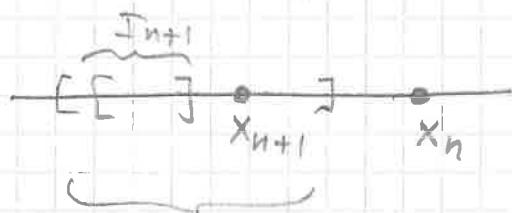
Say,  $\frac{7}{8} \in A_{15}$ . Since  $A_1, \dots, A_{14}$  are finite  
 $\Rightarrow \frac{7}{8}$  will be included in the list (\*) eventually.

So,  $\mathbb{Q} \sim N$  by construction of sets  $A_n$ .

(2) Proof by contradiction. Assume  $R$  is ctable,  
i.e.,  $R \sim N$  and  $\exists f: N \rightarrow R$  (bijection),  
and we can put #'s of  $R$  into a list. Let

$$R = \{x_1, x_2, x_3, x_4, \dots\} \text{ w/ } x_1 = f(1), x_2 = f(2), \dots$$

Let  $I_1$  be a closed interval that does not  
contain  $x_1$ ,  $I_2$  be a closed interval that does not  
contain  $x_2$  and  $I_2 \subset I_1$ . Continue  
in this manner:  $I_{n+1} \subset I_n$  s.t.  $x_{n+1} \notin I_{n+1}$



$$\text{Next: } \bigcap_{n=1}^{\infty} I_n = I_1 \cap I_2 \cap I_3 \cap \dots$$

For any  $x_n$  from the list  $\{x_1, x_2, x_3, \dots\}$ ,

$x_{n_0} \notin I_{n_0}$  and also  $x_{n_0} \notin \bigcap_{n=1}^{\infty} I_n$   
by construction of the intervals. (5)

If  $R = \{x_1, x_2, \dots\}$ , then we must have

$\bigcap_{n=1}^{\infty} I_n = \emptyset$ , but this contradicts the Nested

Interval Property asserting that  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

That is, there is at least one  $x \in R$  which is not on the list  $\{x_1, x_2, \dots\}$ . Thus,

an enumeration of  $R$  is impossible  $\Rightarrow R$  is uncountable.  $\square$

Note:  $\mathbb{N}$  and  $\mathbb{Q}$  are ctable in  $\mathbb{R}$  (uncountable!)

What about  $\mathbb{I}$ ? Since  $R = \mathbb{Q} \cup \mathbb{I}$  it follows that  $\mathbb{I}$  cannot be countable, because otherwise,  $R$  would be. (So,  $\mathbb{I}$  outnumbers  $\mathbb{Q}!$ )

Fact:  $A \cup B$  is ctable if  
 $\begin{matrix} \swarrow & \searrow \\ \text{ctble} & \text{ctble} \end{matrix}$

Other facts:

Thm. (1.5.7) If  $A \subset B$  and  $B$  is countable, then  $A$  is either countable or finite.

Thm. (1.5.8)

A union of countable sets is countable.

$$\bigcup_{i=1}^n A_i \quad \longrightarrow \quad \bigcup_{n=1}^{\infty} A_n$$

(finite union) (infinite)

$$(\aleph_1 = 2^{\aleph_0})$$

Remark:  $\text{card}(R) = \mathfrak{c}$  (continuum) ( $\text{card}(\mathbb{N}) = \aleph_0$ )

"aleph-one" =  $\aleph_1 = \mathfrak{c} > \aleph_0$

"Aleph-null"  
smallest inf. card. #