

## § 1.6 Cantor's Theorem.

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### Diagonalization Method.

Cantor:  $\mathbb{R}$  is uncountable - 1874. (Thm. 1.5.6, (2))

Another proof in 1891. Uses decimal representations of real numbers.

$\exists$  bijection ↴

Idea: Show  $(0,1)$  is uncountable. Then  $\mathbb{R} \sim (0,1)$ .

Theorem (1.6.1) The open interval  $(0,1)$  is uncountable.

Proof: By contradiction. Assume that  $(0,1)$  is countable. Then  $\exists f: \mathbb{N} \rightarrow (0,1)$  that is a bijection.

$\forall n \in \mathbb{N}, f(n) \in (0,1)$  and

$$f(n) = 0. a_{n_1} a_{n_2} a_{n_3} a_{n_4} \dots \quad \text{with } a_{n_k} \in \{0, 1, \dots, 9\}$$

Consider the following array:

$\mathbb{N}$	$(0,1)$
1	$\leftrightarrow f(1) = 0. (a_{11}) a_{12} a_{13} a_{14} a_{15} a_{16} \dots$
2	$\leftrightarrow f(2) = 0. a_{21} (a_{22}) a_{23} a_{24} a_{25} a_{26} \dots$
3	$\leftrightarrow f(3) = 0. a_{31} a_{32} (a_{33}) a_{34} a_{35} a_{36} \dots$
4	$\leftrightarrow f(4) = 0. a_{41} a_{42} a_{43} (a_{44}) a_{45} a_{46} \dots$
5	$\leftrightarrow f(5) = 0. a_{51} a_{52} a_{53} a_{54} (a_{55}) a_{56} \dots$
6	$\leftrightarrow f(6) = 0. a_{61} a_{62} a_{63} a_{64} a_{65} (a_{66}) \dots$
:	:
:	:
:	:
:	:
:	:

All real #'s from  $(0,1)$  are assumed to be on the list.

Now define a number  $x \in (0,1)$  with (2)  
 $x = 0.b_1 b_2 b_3 b_4 \dots$  using the following rule:

$$b_n = \begin{cases} 2, & a_{nn} \neq 2 \\ 3, & a_{nn} = 2 \end{cases}$$

For example:  $a_{11} = 2, a_{22} = 4, a_{33} = 1, a_{44} = 2, \dots$   
 $\Rightarrow x = 0.3223\dots$

Hence,  $x \neq f(1)$  since  $b_1 \neq a_{11}$  (differ in the 1st decimal place). Similarly,  $b_2 \neq a_{22}$ . In general,  $x \neq f(n)$  since  $b_n \neq a_{nn}$ . Thus,  $x$  is not in the range of  $f$  which contradicts the assumption that  $f$  is onto. We conclude that  $(0,1)$  must be uncountable. □

Note: wants to

① If one uses a similar proof to show that  $\mathbb{Q}$  is uncountable:

- yes, we can construct  $x$  in the same way, and  $x$  will fail to be in the range of  $f$ , but  $x$  is not expected to be rational! The decimal expansions of  $\mathbb{Q}$  either terminate or repeat  
 $\Rightarrow$  not true for the constructed  $x$ .

②  $\frac{1}{2} = 0.\overline{5} \stackrel{\text{or}}{=} 0.4999\dots$  Doesn't this cause problems?

- by using digits 2 and 3 for  $x$ , we eliminate the possibility that  $x = 0.b_1 b_2 \dots$  has another possible representation.

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## Power Sets

- Given a set  $A$ , the power set of  $A$ ,  $P(A)$  (or  $2^A$ ) is the set of all subsets of  $A$ .
- Example:  $A = \{\circlearrowleft, \square, \Delta\} \Rightarrow P(A)$  has  $2^3$  elements:  
 $\{\emptyset, \{\circlearrowleft\}, \{\Delta\}, \{\square\}, \{\circlearrowleft, \Delta\}, \{\circlearrowleft, \square\}, \{\Delta, \square\}, A\}$   
↑  
always  
include!
- Fact: If  $A$  is finite w/  $n$  elements, then  $P(A)$  has  $2^n$  elements (proof by induction).
- $A = \{\circlearrowleft, \square, \Delta\} \xrightarrow{?} P(A)$   
 $f: \begin{cases} \circlearrowleft \rightarrow \{\circlearrowleft\} \\ \Delta \rightarrow \{\circlearrowleft, \Delta\} \\ \square \rightarrow \{\circlearrowleft, \Delta, \square\} \end{cases}$  or  $g: \begin{cases} \circlearrowleft \rightarrow \{\Delta, \square\} \\ \Delta \rightarrow \emptyset \\ \square \rightarrow \{\circlearrowleft, \square\} \end{cases} \dots \quad \left. \begin{array}{l} \text{but} \\ \text{not onto} \end{array} \right]$
- Note:  $2^n > n + n$  (can you show this?),   
 $P(A)$  has too many elements to allow onto mappings between  $A$  and  $P(A)$ . ↙ (A-finite or infinite)
- Theorem (1.6.2): Cantor's Theorem  
Given any set  $A$ , there does not exist a function  $f: A \rightarrow P(A)$  that is onto.
- Proof: Assume, for contradiction, that   
 $\exists f: A \rightarrow P(A)$  which is onto. Thus,  $\forall a \in A$ ,   
 $f(a)$  is a subset of  $A$ , and  $\nexists$  subset of A elem. of  $P(A)$   
 $\exists a \in A$  s.t.  $f(a)$  is this subset.
- Consider  $\forall a \in A$  and  $f(a) \in P(A)$ . If  $a \notin f(a)$ , then let  $B = \{a \in A : a \notin f(a)\}$ . Since  $f$  is onto,

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then  $B = f(A')$  for some  $A' \subseteq A$ . First, if

$a' \in B \Rightarrow a' \notin f(a')$        $\stackrel{\text{by def}}{\text{in } B} \quad \stackrel{\text{not in } B}{\text{not in } f(a')}$

$\Rightarrow a' \notin B \Rightarrow$  a contradiction. Next, if  $a' \notin B \Rightarrow$   
 by def. of  $B$ ,  $a' \in f(a')$ , but since  $B = f(A') \Rightarrow$   
 $a' \in B \Rightarrow$  another contradiction. Hence,  $a'$  cannot  
 be in neither  $B$  nor  $B^c \Rightarrow \nexists a' \text{ s.t. } f(a') = B \Rightarrow$   
 $f: A \rightarrow P(A)$  is not onto.

Subset of  $A$   
 that is not  
 equal  $f(a)$   
 for any  $a \in A$

Cantor's Theorem  $\Rightarrow \forall A, A \neq P(A)$

(In particular,  $\mathbb{N} \neq P(\mathbb{N})$ )

Fact:  $P(\mathbb{N}) \sim \mathbb{R}$

The following sets have different cardinalities:

$\underbrace{\mathbb{N}}$ ,  $\underbrace{P(\mathbb{N})}$ ,  $\underbrace{P(P(\mathbb{N}))}$  ...  
 ↓            ↓            ↓  
 ctable        unctble!  
 (also,  $\mathbb{Q}$  &  $\mathbb{Z}$ )

different classes of cardinality

From  
 Cantor's  
 Thm.       $\left\{ \begin{array}{l} \mathbb{N} \subset \mathbb{R}, \text{ card } \mathbb{N} < \text{card } \mathbb{R} \\ \forall A, \text{ card } A < \text{card } P(A) \end{array} \right.$