

§2.2 The Limit of a Sequence.

(1)

Def: A sequence is a function whose domain is \mathbb{N} .

Given $f: \mathbb{N} \rightarrow \mathbb{R}$, $x_n = \underbrace{f(n)}_{n\text{th term on the list}}$

Examples:

$$(1) \quad x_n = n, \text{ i.e., } (1, 2, 3, \dots)$$

$$(2) \quad x_n = \frac{1}{n}, \text{ i.e., } (1, \frac{1}{2}, \frac{1}{3}, \dots)$$

$$(3) \quad a_n = 2^n, \text{ i.e., } (2, 4, 8, \dots)$$

$$(4) \quad (x_n), \text{ where } x_1 = 2 \text{ and } x_{n+1} = \frac{x_n + 1}{2}$$

Note: 1) sometimes the starting index can be $n=0$ (or $n=n_0 \neq 1$).

2) a sequence is an infinite list of #'s,

Def: Convergence of a Sequence.

A sequence (a_n) converges to a real number a if $\forall \varepsilon > 0 \exists \underset{\substack{\uparrow \\ N=N(\varepsilon)}}{N \in \mathbb{N}} \text{ s.t. } \forall n \geq N \Rightarrow |a_n - a| < \varepsilon$.
distance

We write: $\lim_{n \rightarrow \infty} a_n = a \quad \left(\underset{n \rightarrow \infty}{(a_n \rightarrow a)} \right)$
 ↓
 limit of the sequence.

Def: Let $a \in \mathbb{R}$ and $\varepsilon > 0$, then the set

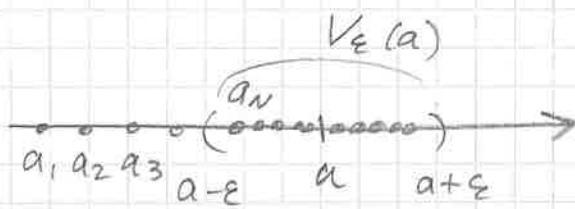
$V_\varepsilon(a) = \{x \in \mathbb{R} : |x - a| < \varepsilon\}$ is called
 the ε -neighborhood of a $a - \varepsilon < x < a + \varepsilon$



Def: Convergence Revisited.

(2)

A sequence (a_n) converges to $a \in \mathbb{R}$ if for any ϵ -neighborhood $V_\epsilon(a)$ of a , there exists a pt in (a_n) after which all terms are in $V_\epsilon(a)$.
 ($\forall V_\epsilon(a)$ contains all but finite # terms of (a_n))



Examples :

(1) $a_n = \frac{1}{n}$. Prove that $\lim_{n \rightarrow \infty} a_n = 0$

$$\parallel \forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \Rightarrow \left| \frac{1}{n} - 0 \right| < \epsilon$$

or, simply, $\frac{1}{n} < \epsilon$. \leftarrow want possible!

Proof: Let $\epsilon > 0$. Choose $N > \frac{1}{\epsilon}$ (by the Archimedean prop.). Set $n \geq N$. Then,

$$n \geq N > \frac{1}{\epsilon} \Rightarrow \frac{1}{n} < \epsilon \Rightarrow \left| \frac{1}{n} - 0 \right| < \epsilon. \square$$

(Note : we wanted

$$\begin{aligned} & \frac{1}{n} < \epsilon \\ & \updownarrow \text{equivalent} \\ & \frac{1}{\epsilon} < n \\ & \underbrace{\quad}_{\text{choose } N > \frac{1}{\epsilon}} \end{aligned}$$

(2) $a_n = \frac{1}{\sqrt{n}}$. Prove that $\lim_{n \rightarrow \infty} a_n = 0$.

Proof: Let $\epsilon > 0$. We want $\left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon$ or

$$\frac{1}{\sqrt{n}} < \epsilon, \text{ which is equivalent to } \frac{1}{\epsilon} < \sqrt{n}$$

or $\frac{1}{\epsilon^2} < n$. So, choosing $N > \frac{1}{\epsilon^2}$, we get:

$$\forall \epsilon > 0 \ \exists N > \frac{1}{\epsilon^2} \text{ s.t. } \forall n \geq N > \frac{1}{\epsilon^2} \Rightarrow \frac{1}{\sqrt{n}} < \epsilon \Rightarrow \left| \frac{1}{\sqrt{n}} - 0 \right| < \epsilon \quad \square$$

- Template for a proof that $\lim_{n \rightarrow \infty} (a_n) = a$

(3)

- 1) Let $\epsilon > 0$ (arbitrary)
- 2) Choose $N \in \mathbb{N}$ ← requires most work, usually done before
- 3) Show that the chosen N works ←
- 4) Assume $n \geq N$
- 5) 4) implies $|a_n - a| < \epsilon$

More Examples:

(3) Show that $\lim_{n \rightarrow \infty} \frac{n-1}{n+1} = 1$ (sequence $(\frac{n-1}{n+1})$)

Before writing a formal proof, let us do some work: we want $\left| \frac{n-1}{n+1} - 1 \right| < \epsilon$ for all $n \geq N(\epsilon)$

$$\left| \frac{n-1}{n+1} - 1 \right| = \left| \frac{(n-1) - (n+1)}{n+1} \right| = \frac{2}{n+1} < \epsilon \iff$$

$$\frac{2}{\epsilon} < n+1 \quad \text{or} \quad n > \frac{2}{\epsilon} - 1 \quad \text{So, choosing } N > \frac{2}{\epsilon} - 1$$

should work!

Proof: let $\epsilon > 0$, choose $N \in \mathbb{N}$, $N > \frac{2}{\epsilon} - 1$

Then $\forall n \in \mathbb{N}, n \geq N > \frac{2}{\epsilon} - 1$, we have

$$\left| \frac{n-1}{n+1} - 1 \right| < \epsilon.$$

□

(4) See Example 2.2.6 in the text.

- Theorem (2.2.7) Uniqueness of Limits.

The limit of a sequence, when it exists, must be unique.

Proof: Start from $\lim_{n \rightarrow \infty} (a_n) = a$ & $\lim_{n \rightarrow \infty} (a_n) = b$
and show $a = b$. → HW exercise.

- Divergence

Def: A sequence that does not converge is said to diverge.

Examples:

- { 1) $x_n = n$, i.e. $(1, 2, 3, 4, \dots)$ ← unbounded
 - { 2) $x_n = (-1)^n$, i.e. $(-1, 1, -1, 1, -1, 1, \dots)$ ← oscillating between ± 1
- no single number satisfying def. of convergence.

(Discuss more later)

- 3) To show that $\lim_{n \rightarrow \infty} x_n \neq x$, we must find $\epsilon > 0$ for which no N works (s.t. $|x_n - x| < \epsilon$), i.e.:
 $\exists \epsilon > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists n > N \Rightarrow |x_n - x| \geq \epsilon$.

See Example 2.2.8