

## § 2.3. The Algebraic & Order Limit Theorems.

(Several important results)

Def: A sequence  $(x_n)$  is bounded if there exists a number  $M > 0$  s.t.  $|x_n| \leq M \quad \forall n \in \mathbb{N}$ .

Geometrically:  $(x_n)$  is in the interval  $[-M, M]$ .



Theorem (2.3.2). Every convergent sequence is bounded.

Proof: Let  $\lim_{n \rightarrow \infty} x_n = a$ , i.e.  $\forall \epsilon > 0 \exists N \in \mathbb{N}$  s.t.  $\forall n \geq N \Rightarrow |x_n - a| < \epsilon$ .

$$\epsilon = 1: \frac{(a_1 - a)}{a_1 + 1}$$

If we take  $\epsilon = 1$ , then  $\exists N \in \mathbb{N}$  s.t. if  $n \geq N$ , then  $x_n \in V_1(a) = (a-1, a+1)$ . Note  $|x_n| < |a| + 1 \quad \forall n \geq N$

Now let  $M = \max \{ |x_1|, |x_2|, \dots, |x_{N-1}|, |a| + 1 \}$

Thus,  $|x_n| < M$  for all  $n \in \mathbb{N}$ . □

Note: Bounded  $\not\Rightarrow$  convergent

( $x_n = (-1)^n$  is bounded but not convergent)

However, if a sequence is not bounded, then it cannot be convergent (as otherwise it would be bounded).

## Theorem (2.3.3) Algebraic Limit Theorem.

Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then,

$$(1) \lim_{n \rightarrow \infty} (ca_n) = ca \quad \forall c \in \mathbb{R}$$

$$(2) \lim_{n \rightarrow \infty} (a_n + b_n) = a + b$$

$$(3) \lim_{n \rightarrow \infty} (a_n b_n) = ab$$

$$(4) \lim_{n \rightarrow \infty} \left( \frac{a_n}{b_n} \right) = \frac{a}{b} \quad b \neq 0$$

Proof:

(1) If  $c=0$ , then the sequence  $(0a_n) = (0, 0, 0, \dots)$  clearly converges to 0.

Let  $c \neq 0$ , then to show that  $\lim_{n \rightarrow \infty} ca_n = ca$ , we take  $\epsilon > 0$  and we want

$$|ca_n - ca| < \epsilon \quad \text{for some } N \text{ and } n \geq N.$$

$|ca_n - ca| = |c||a_n - a|$ . Since  $a_n \xrightarrow{n \rightarrow \infty} a$ , then for this  $\epsilon$ , we can choose  $N$  s.t.  $|a_n - a| < \frac{\epsilon}{|c|}$  for  $n \geq N$ . This  $N$  works for  $ca_n$  as well:

$$\begin{aligned} \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \text{ s.t. } \forall n > N, \quad |ca_n - ca| &= |c||a_n - a| \\ &< |c| \frac{\epsilon}{|c|} = \epsilon. \end{aligned} \quad \square$$

(2) Now let  $a_n \xrightarrow{n \rightarrow \infty} a$  and  $b_n \xrightarrow{n \rightarrow \infty} b$ . Let  $\epsilon > 0$ .

Then we can find  $N_a$  and  $N_b$  s.t.  $|a_n - a| < \frac{\epsilon}{2}$  for  $n \geq N_a$  and  $|b_n - b| < \frac{\epsilon}{2}$  for  $n \geq N_b$ .

By choosing  $N = \max\{N_a, N_b\}$  and letting  $n \geq N$ , we have:  $|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)|$

$$\leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \text{Hence,} \\ \begin{matrix} \text{triangle} \\ \text{inequality} \end{matrix} \quad a_n + b_n \xrightarrow{n \rightarrow \infty} a + b. \quad \square$$

(3) To show  $a_n b_n \xrightarrow{n \rightarrow \infty} ab$ , we first observe that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - \underbrace{a b_n + a b_n - ab}_0| \leq \\ &\leq |a_n b_n - a b_n| + |a b_n - ab| \\ &= |b_n| |a_n - a| + |a| |b_n - b| \end{aligned}$$

Note: since  $b_n \xrightarrow{n \rightarrow \infty} b$ , then  $\exists M > 0$  s.t.  $|b_n| \leq M$

$\forall n \in \mathbb{N}$ . Also, since  $b_n \xrightarrow{n \rightarrow \infty} b$ ,  $a_n \xrightarrow{n \rightarrow \infty} a$ ,  $\forall \epsilon > 0$  we can find  $N_b$  s.t.  $\forall n \geq N_b$ ,  $|b_n - b| < \frac{\epsilon}{M+|a|}$

and  $N_1$  s.t.  $\forall n \geq N_1$ ,  $|a_n - a| < \frac{\epsilon}{M+|a|}$ . (3)

Now, choosing  $N = \max\{N_1, N_2\}$ , we can make

$$\begin{aligned} |a_n b_n - ab| &\leq |b_n| |a_n - a| + |a| |b_n - b| \\ &< M \frac{\epsilon}{M+|a|} + |a| \frac{\epsilon}{M+|a|} = \frac{M+|a|}{M+|a|} \epsilon = \epsilon. \end{aligned}$$

So,  $a_n b_n \xrightarrow[n \rightarrow \infty]{} ab$ . □

(4) Finally, let us prove  $\frac{a_n}{b_n} \xrightarrow[n \rightarrow \infty]{} \frac{a}{b}$  ( $b \neq 0$ )

Note that this follows from (3)  
if we prove that  $\frac{1}{b_n} \xrightarrow[n \rightarrow \infty]{} \frac{1}{b}$ .

$$\text{Observe that } \left| \frac{1}{b_n} - \frac{1}{b} \right| = \frac{|b - b_n|}{|b| |b_n|}$$

So  $|b - b_n|$  can be made as small as we want by choosing  $n$  large. What about  $\frac{1}{|b| |b_n|}$ ?

In particular,  $\frac{1}{|b_n|}$  for large  $n$ ?

Pick  $\epsilon = \frac{|b|}{2}$  then  $\exists N \in \mathbb{N}$  s.t.  $\forall n \geq N$ ,  $|b_n - b| < \frac{|b|}{2}$ .

$$\text{So, } |b| - |b_n| \leq |b - b_n| < \frac{|b|}{2} \Rightarrow$$

Exer.  
1.2.6(d)  $\leftarrow$  from  $\overbrace{|x - y|}^{\leq |x - y|} \leq |x - y| \quad (\forall x, y \in \mathbb{R}) \rightarrow \frac{1}{|b_n|} < \frac{2}{|b|}$

$$|b| - |b_n| < \frac{|b|}{2} \Rightarrow |b_n| > \frac{|b|}{2} \text{ for } n \geq N.$$

Choosing  $N_2$  s.t.  $\forall n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon |b|^2}{2}$ , we get:

$\forall \epsilon > 0 \exists N = \max\{N_1, N_2\}$  s.t.  $\forall n \geq N$ ,

$$(*) \quad \left| \frac{1}{b_n} - \frac{1}{b} \right| = |b - b_n| \frac{1}{|b| |b_n|} < \frac{\epsilon |b|^2}{2} \frac{1}{|b|} \frac{2}{|b|} = \epsilon.$$

(\*) & (3) imply  $\frac{a_n}{b_n} \xrightarrow[n \rightarrow \infty]{} \frac{a}{b}$  as  $n \rightarrow \infty$ . □

(4)

### Theorem 2.3.4 (Order Limit Theorem)

Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ .

(1) If  $a_n \geq 0 \ \forall n \in \mathbb{N} \Rightarrow a \geq 0$

(2) If  $a_n \leq b_n \ \forall n \in \mathbb{N} \Rightarrow a \leq b$

(3) If  $\exists c \in \mathbb{R}$  w/  $c \leq b_n \ \forall n \in \mathbb{N}$ , then  $c \leq b$ .

(Similarly, if  $a_n \leq c \ \forall n \in \mathbb{N}$ , then  $a \leq c$ )

Proof:

(1) By contradiction: assume that  $a < 0$ . Let  $\epsilon = |a|$ , then  $\exists N$  s.t.  $|a_n - a| < |a| \ \forall n \geq N$ . Hence,

$$|a_n - a| < |a| \Rightarrow -|a| < a_n - a < |a| \Rightarrow$$

$a_n < |a| + a = -a + a = 0$ . This is a contradiction w/ the hypothesis that  $a_n \geq 0 \ \forall n \in \mathbb{N}$ . So,  
 $a \geq 0$ .  $\square$

(2) By the previous thm (2.3.3),  $\lim_{n \rightarrow \infty} (b_n - a_n) = b - a$ .

Since  $b_n - a_n \geq 0 \ \forall n \in \mathbb{N} \Rightarrow$  by (1),  $b - a \geq 0$  and  $b \geq a$ .  $\square$

(3) Given  $c \in \mathbb{R}$  s.t.  $b_n \geq c \ \forall n \in \mathbb{N}$ , consider  $a_n = c \ \forall n \in \mathbb{N}$ , then by (2),  $c \leq b$ .  $\square$

Remark: The theorem can be "weakened" by requiring that

(1)  $a_n \geq 0 \ \forall n \geq N_a$  for some  $N_a$

(2)  $a_n \leq b_n \ \forall n \geq N$  for some  $N$

(3)  $c \leq b_n \ \forall n \geq N_b$  for some  $N_b$

limits and their properties do not depend on what happens at the beginning of the sequence!