

§ 2.4The Monotone Convergence Theorem. ①
Infinite Series.

Recall: if $a_n \rightarrow a \Rightarrow a_n$ is bdd.

The converse is $\overset{n \rightarrow \infty}{\text{not}}$ always true, but if a sequence is monotone \Rightarrow it converges.

Def: A sequence (a_n) is ^(↑) increasing if $a_n \leq a_{n+1} \forall n \in \mathbb{N}$ and ^(↓) decreasing if $a_n \geq a_{n+1} \forall n \in \mathbb{N}$. In either case, it is called monotone.

Ex: $\begin{cases} x_n = \frac{1}{n} \text{ is decreasing: } (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots) \\ x_n = n \text{ is increasing: } (1, 2, 3, 4, \dots) \end{cases}$

Theorem (2.4.2) Monotone Convergence Theorem.

If a sequence is monotone and bounded, it converges.

a_n bdd above $\Rightarrow \exists \sup$ by AoC

In particular: $a_n \uparrow$ and $s = \sup \{a_n\} \Rightarrow \lim_{n \rightarrow \infty} a_n = s$; similarly, $b_n \downarrow$ and $i = \inf \{b_n\} \Rightarrow$

$\lim_{n \rightarrow \infty} b_n = i$.

b_n bdd below $\Rightarrow \exists \inf$
- comes from Exer. 1.33.

Proof: Let $a_n \uparrow$ and $s = \sup \{a_n\}$. Take $\epsilon > 0$.

Note that $s - \epsilon$ is not an u.b., so arbitrary

$\exists a_N$ s.t. $s - \epsilon < a_N$. Since $a_n \uparrow$, then if $n \geq N$, $a_n \geq a_N$, and $s - \epsilon < a_N \leq a_n \leq s < s + \epsilon$ or $|a_n - s| < \epsilon$. That is, $\forall \epsilon > 0 \exists N$ s.t. $\forall n \geq N$, $|a_n - s| < \epsilon$. Thus, $\lim_{n \rightarrow \infty} a_n = s = \sup \{a_n\}$. \square

(The case w/ $b_n \downarrow$ and bdd below is proven similarly \rightarrow try it.)

Note: if $x_n \nearrow (\searrow)$ and is not bounded above (below) then $\lim_{n \rightarrow \infty} x_n = \infty$ ($\lim_{n \rightarrow \infty} x_n = -\infty$), i.e. $\forall E > 0 \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow x_n > E$ (DIY for $x_n \rightarrow -\infty$) (2)

* Thm. above is important for the study of infinite series.

Def: Let (x_n) be a sequence. An infinite series is $\sum_{n=1}^{\infty} x_n = x_1 + x_2 + x_3 + \dots$, and the corresponding sequence of partial sums (S_m) is $S_m = \underbrace{x_1 + x_2 + \dots + x_m}_{\text{first } m \text{ terms}}$

We say $\sum_{n=1}^{\infty} x_n$ converges to S , if the sequence (S_m) converges to S . $\left(\sum_{n=1}^{\infty} x_n = S \right)$

Examples

$$(1) \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots, \quad S_m = \underbrace{1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2}}_{> 0}$$

Note: $S_m \nearrow$, but is it bounded above?

$$S_m = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2} = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \dots + \frac{1}{m \cdot m} <$$

$$< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \dots + \frac{1}{m(m-1)}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{m-1} - \frac{1}{m}\right)$$

$$= 1 + 1 - \frac{1}{m} = 2 - \frac{1}{m} < 2. \text{ Thus, } S_m \nearrow \text{ and bdd}$$

above $\Rightarrow S_m$ converges to some S and

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = S \text{ (found later!)}$$

(2) Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ u/l

$$S_m = 1 + \underbrace{\frac{1}{2}}_{S_1} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{S_2} + \dots + \underbrace{\frac{1}{m}}_{S_3 \dots}$$

$$\begin{aligned} S_2 &= 1 + \frac{1}{2} \\ S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \end{aligned}$$

see p. 58

$S_m \nearrow$. Bounded? Nope: $S_{2^k} > 1 + \frac{k}{2} \xrightarrow{(k \geq 2)}$
 for large k , S_{2^k} grows. Thus, S_m diverges \Rightarrow
 $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $((S_{2^k}))$ is a subsequence of (S_m)

Theorem (2.4.6)

Cauchy Condensation Test (CCT)

Augustin-Louis Cauchy, 1789-1857,
 French mathematician, engineer,
 and scientist

Suppose (x_n) is decreasing and $x_n \geq 0 \forall n \in \mathbb{N}$.

Then, the series $\sum_{n=1}^{\infty} x_n$ converges iff the series
 $\sum_{n=0}^{\infty} 2^n x_{2^n} = x_1 + 2x_2 + 4x_4 + 8x_8 + 16x_{16} + \dots$ converges.

Proof:

(\Rightarrow) Let us show that $\sum_{n=1}^{\infty} x_n$ converges \Rightarrow
 $\sum_{n=0}^{\infty} 2^n x_{2^n}$ converges by contrapositive, i.e. if
 $\sum_{n=0}^{\infty} 2^n x_{2^n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} x_n$ diverges.

Let S_m be partial sum of $\sum_{n=1}^{\infty} x_n$ and
 \bar{S}_m be partial sum of $\sum_{n=0}^{\infty} 2^n x_{2^n}$. Since
 $\sum_{n=0}^{\infty} 2^n x_{2^n}$ diverges $\Rightarrow (\bar{S}_m)$ diverges $\Rightarrow (\bar{S}_m)$ unbdd.

Let us show that (S_m) is also not bdd
 $(\Rightarrow \sum_{n=1}^{\infty} x_n$ diverges). Recall: $x_n \downarrow$. We have:

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$$\begin{aligned}
 2S_{2^m} &= 2x_1 + 2x_2 + 2x_3 + 2x_4 + \dots + 2x_{2^{m-1}+1} + \\
 &+ 2x_{2^{m-1}+2} + \dots + 2x_{2^m} = 2x_1 + 2x_2 + (2x_3 + 2x_4) + \dots \\
 &+ \underbrace{(2x_{2^{m-1}+1} + 2x_{2^{m-1}+2} + \dots + 2x_{2^m})}_{\text{total } 2^{m-1} \text{ terms}} \geq \underbrace{\text{total } 2^{m-1} \text{ terms}}_{x_n \downarrow} \\
 &\geq 2x_1 + 2x_2 + (2x_3 + 2x_4) + \dots + (2x_{2^m} + 2x_{2^m} + \dots + 2x_{2^m}) \\
 &= 2x_1 + 2x_2 + 4x_3 + 8x_4 + \dots + 2^m x_{2^m} \geq \\
 &\geq x_1 + 2x_2 + 4x_3 + 8x_4 + \dots + 2^m x_{2^m} = \overline{S_m}.
 \end{aligned}$$

Since $(\overline{S_m})$ is not bdd \Rightarrow (from above)
 (S_{2^m}) not bdd $\Rightarrow (\overline{S_m})$ is not bdd $\Rightarrow \sum_{n=1}^{\infty} x_n$
 diverges!

(\Leftarrow) Assume $\sum_{n=0}^{\infty} 2^n x_{2^n}$ converges. Show
 that $\sum_{n=1}^{\infty} x_n$ converges too.

$(\overline{S_m})$ bdd (by Thm. 2.3.2)

$\overline{S_m}$ (partial sum)

$\overline{S_m} = x_1 + 2x_2 + 4x_3 + \dots + 2^m b_{2^m}$ is bdd \Rightarrow
 $\exists M > 0$ s.t. $|\overline{S_m}| \leq M \quad \forall m \in \mathbb{N}$. Recall $x_n \geq 0$
 $\forall n \in \mathbb{N} \Rightarrow S_m = x_1 + x_2 + \dots + x_m \nearrow$. We need to
 show (S_m) is bdd.

Consider m and k s.t. $m \leq 2^{k+1}-1$. Then

$S_m \leq S_{2^{k+1}-1}$ (since $S_m \nearrow$) and

$$\begin{aligned}
 S_{2^{k+1}-1} &= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + \dots + (x_{2^k} + \dots + x_{2^{k+1}-1}) \\
 &\leq x_1 + (x_2 + x_2) + (x_4 + x_4 + x_4 + x_4) + \dots + (x_{2^k} + \dots + x_{2^k}) \\
 &\leq x_1 + 2x_2 + 4x_4 + \dots + 2^k x_{2^k} = \overline{S_k}. \text{ Since}
 \end{aligned}$$

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$$S_m \leq S_{2^{k+1}-1} \leq \overline{S_k} \leq M \Rightarrow S_m \text{ is bdd}$$

Recall S_m is also increasing. Thus, by the Monotone Convergence Thm, $\sum_{n=1}^{\infty} x_n$ converges. \square

Corollary (2.4.7)

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

(Later! \rightarrow need to learn about geom. series)

Examples :

① Back to $\sum_{n=1}^{\infty} \frac{1}{n}$. So, $x_n = \frac{1}{n} > 0 \quad \forall n \in \mathbb{N}$

and $\frac{1}{n} \downarrow$. Notice that $x_{2^n} = \frac{1}{2^n}$ and

$$\sum_{n=0}^{\infty} 2^n x_{2^n} = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2^n} \right) = \sum_{n=0}^{\infty} 1 = 1 + 1 + 1 + \dots \text{ diverges}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges too.} \quad \nearrow, \text{ unbdd}$$

② Consider $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$.

$$x_n = \frac{1}{n \ln n} > 0 \quad \forall n \geq 2 \quad \text{and } x_n \downarrow$$

$$x_{2^n} = \frac{1}{2^n \ln(2^n)} = \frac{1}{n 2^n \ln 2} \Rightarrow$$

$$\sum_{n=1}^{\infty} 2^n x_{2^n} = \sum_{n=1}^{\infty} 2^n \frac{1}{n 2^n \ln 2} = \sum_{n=1}^{\infty} \frac{1}{n \ln 2}$$

$$= \frac{1}{\ln 2} + \frac{1}{2 \ln 2} + \frac{1}{3 \ln 2} + \dots = \underbrace{\frac{1}{\ln 2}}_{\text{constant}} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{diverges by CT}}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \ln(n)} \text{ diverges too.} \quad \text{see ①}$$

③ DIY: Show that $\sum_{n=2}^{\infty} \frac{1}{n \ln(n^3)}$ diverges.

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More Examples :

Exer. 2.4.8 (c) : Converges or diverges?

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right), \text{ i.e. } x_n = \ln\frac{n+1}{n} = \ln(n+1) - \ln n.$$

$$\begin{aligned} \text{Then } S_m &= \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + \ln\left(\frac{4}{3}\right) + \dots + \ln\left(\frac{m+1}{m}\right) \\ &= (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \dots + (\ln m - \ln(m-1)) + \\ &\quad + (\ln(m+1) - \ln m) = -\ln 1 + \ln(m+1) = \ln(m+1) \end{aligned}$$

As $m \rightarrow \infty$, S_m grows without bound.

Thus, $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ diverges.

One more :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}. \text{ Converges or diverges?}$$

$$\begin{aligned} \text{if } S_1 &= \frac{1}{2}, S_2 = \frac{1}{2} + \frac{1}{6}, S_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12}, S_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\ S_{n+1} &= S_n + \frac{1}{(n+1)(n+2)} \end{aligned}$$

Can prove that $S_n = 1 - \frac{1}{n+1}$. By induction:

$$S_1 = 1 - \frac{1}{2} = \frac{1}{2} \quad \checkmark$$

Suppose $S_n = 1 - \frac{1}{n+1}$ and let us show

$$S_{n+1} = 1 - \frac{1}{n+2}.$$

$$S_{n+1} = S_n + \frac{1}{(n+1)(n+2)} = S_n + \underbrace{\left(\frac{1}{n+1} - \frac{1}{n+2}\right)}$$

$$= \left(1 - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \quad \text{partial fractions}$$

$$= 1 - \frac{1}{n+2}. \quad \text{Thus, } S_n = 1 - \frac{1}{n+1} \text{ and}$$

$$\lim_{n \rightarrow \infty} S_n = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \text{ converges (to 1).}$$