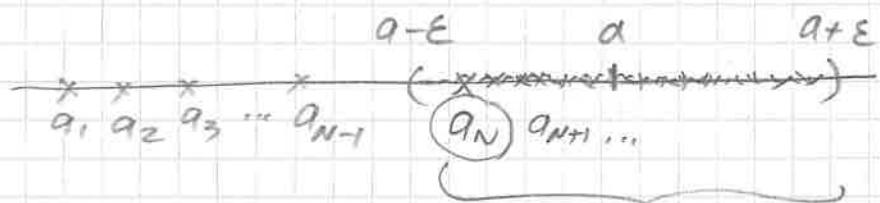


- Def. A sequence (a_n) is called a Cauchy sequence if, $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \Rightarrow |a_n - a_m| < \varepsilon$.
 - Recall: Def. of convergence of a sequence to a : $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$
 $(\lim_{n \rightarrow \infty} a_n = a)$
- Geometrically: given $\varepsilon > 0$, it is possible to find a_N s.t. all the terms after $(N=N(\varepsilon))$ are within ε -neighborhood of a :



- For a Cauchy sequence: Given $\varepsilon > 0$, there exists N th term $(N=N(\varepsilon))$, after which all the terms are closer to each other than ε .

It turns out that :

- (1) convergent sequences are Cauchy sequences.
- (2) Cauchy sequences converge.

- Theorem (2.6.2) Every convergent sequence is a Cauchy sequence.

Proof: Let $a_n \xrightarrow{n \rightarrow \infty} a$. Then $\forall \varepsilon > 0 \exists N$ s.t. $\forall n \geq N \Rightarrow |a_n - a| < \frac{\varepsilon}{2}$, and $\forall m \geq N \Rightarrow |a_m - a| < \frac{\varepsilon}{2}$.

$|a_m - a| < \frac{\epsilon}{2}$. That is, $\forall \epsilon > 0, \exists N$ s.t. (2)

$$\begin{aligned} \forall m, n \geq N, \quad |a_n - a_m| &= |(a_n - a) + (a - a_m)| \\ &= |(a_n - a) + (a - a_m)| \leq |a_n - a| + |a_m - a| \\ &\stackrel{\text{triangle}}{\leq} |a_n - a| + |a_m - a| \\ &\stackrel{\text{inequality}}{\leq} \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, (a_n) is a Cauchy sequence. \square

- Lemma (2.6.3) Cauchy sequences are bounded.

Proof Let $\epsilon = 1$, then $\exists N$ s.t. $\forall m, n \geq N \Rightarrow |a_n - a_m| < \epsilon \stackrel{(\text{let } m=N)}{\Rightarrow} |a_n - a_N| < \epsilon \Rightarrow |a_n| - |a_N| \leq |a_n - a_N| < 1 \Rightarrow |a_n| < |a_N| + 1 \quad \forall n \geq N$. Then we can choose $M = \max \{|a_1|, |a_2|, \dots, |a_{N-1}|, |a_N| + 1\}$ and, hence $|a_n| < M \quad \forall n \in \mathbb{N}$ and is bounded. \square

- Theorem (2.6.4) Cauchy Criterion

A sequence converges iff it is a Cauchy sequence.

Proof: (\Rightarrow) Theorem (2.6.2) ✓ (see above)

(\Leftarrow) Suppose we have a Cauchy sequence (a_n) . We will use Lemma 2.6.3 & Bolzano-Weierstrass theorem to show the sequence converges.

Lemma 2.6.3 $\Rightarrow (a_n)$ is bounded. By B-W thm, there exists a subseq. (a_{n_k}) w/ $\lim_{k \rightarrow \infty} a_{n_k} = a$.

To show that $\lim_{n \rightarrow \infty} a_n = a$ as well, we will pick $\epsilon > 0$. Then $\exists N$ s.t. $\forall m, n \geq N$,

$\Rightarrow |a_n - a_m| < \frac{\epsilon}{2}$. For the same ϵ and N , let us

(3)

Choose $n_E \geq N$ s.t. $|a_{n_E} - a| < \frac{\varepsilon}{2}$. Now

$$|a_n - a| = |a_n - a_{n_E} + a_{n_E} - a|$$

$$\leq |a_n - a_{n_E}| + |a_{n_E} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

\smile \smile \smile

(if $n \geq N$; $n_E \geq N$)

That is, $\forall \varepsilon > 0 \exists N$ s.t. $\forall n \geq N \Rightarrow |a_n - a| < \varepsilon$,
and hence, $\lim_{n \rightarrow \infty} a_n = a$. \square

Conclusion:

AoC \Rightarrow



(NIP) (BW)
 $\left\{ \begin{array}{l} \text{Nested Int. Prop.} \Rightarrow B\text{-W. Thm} \Rightarrow \\ \text{Monotone Conv. Thm.} \end{array} \right.$ Cauchy
 (MCT) criterion
 (CC)

\forall bdd above nonempty
set of reals has
a l.u.b.

See p. 69 on discussion about connection
between AoC, NIP, BW, MCT, CC and Archimedean
prop.