

§ 2.7

# Properties of Infinite Series.

①

Recall : sequence  $(a_n) = (a_1, a_2, a_3, \dots)$

series  $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$

partial sums  $(s_1, s_2, s_3, \dots)$

where  $s_m = \sum_{n=1}^m a_n$  and

$\lim_{m \rightarrow \infty} s_m = S = \sum_{n=1}^{\infty} a_n$  from

- Translating results from the theory of sequences into the behavior of infinite series.

Theorem (2.7.1) Algebraic Limit Thm for Series

If  $\sum_{k=1}^{\infty} a_k = A$  and  $\sum_{k=1}^{\infty} b_k = B$ , then

(i)  $\sum_{k=1}^{\infty} c a_k = cA$   $\forall c \in \mathbb{R}$

(ii)  $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$ .

Proof : We will prove (i) only.

(i) Let  $s_m = a_1 + a_2 + \dots + a_m$  and  
 $t_m = c a_1 + c a_2 + \dots + c a_m = c s_m$ . By Alg. Lim. Thm  
 for sequences,  $\lim_{m \rightarrow \infty} t_m = \lim_{m \rightarrow \infty} c s_m = c \underbrace{\lim_{m \rightarrow \infty} s_m}_A = cA$ . □

Theorem (2.7.2) Cauchy Criterion for Series.

The series  $\sum_{k=1}^{\infty} a_k$  converges iff  $\forall \epsilon > 0 \exists N \in \mathbb{N}$   
 s.t.  $\forall n > m \geq N$ ,  $|a_{m+1} + \dots + a_n| < \epsilon$  ( $(s_m)$  is Cauchy)

Proof :  $|s_n - s_m| = |a_{m+1} + \dots + a_n|$ . Then

$(s_m)$  converges iff it is a Cauchy seq., i.e.

$\forall \epsilon > 0 \exists N$  s.t.  $\forall n, m \geq N$ ,  $|s_n - s_m| < \epsilon$ . (Thm 2.6.4) □

Theorem (2.7.3) If  $\sum_{k=1}^{\infty} a_k$  converges  $\Rightarrow$

$$\lim_{k \rightarrow \infty} a_k = 0.$$

Proof:  $\sum_{k=1}^{\infty} a_k$  converges  $\Rightarrow \forall \varepsilon > 0 \exists N$  s.t.

$$\forall n > m \geq N \Rightarrow |a_{m+1} + \dots + a_n| < \varepsilon \text{ by Thm. 2.7.2.}$$

If  $n = m+1 \Rightarrow |a_n| < \varepsilon$ . That is,  $\forall \varepsilon > 0$ ,

$\exists N$  s.t.  $\forall n > N \Rightarrow |a_n - 0| < \varepsilon \Rightarrow \lim_{n \rightarrow \infty} a_n = 0$ .  $\square$

Examples:

(1) The converse of Thm. 2.7.3 is not true:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$
 diverges, whereas  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ .

(2) Also:  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges by Cauchy Criterion:  
 $n-m$  terms

$$(n > m) S_n - S_m = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{n} \geq \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$
$$= (n-m) \cdot \frac{1}{n} = \frac{n-m}{n} \Rightarrow \forall m, S_{8m} - S_m \geq \frac{m}{2m} = \frac{1}{2}$$

$\Rightarrow S_m$  can not be Cauchy!

(3)  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges  $\Rightarrow \lim_{n \rightarrow \infty} \frac{1}{n^2} = 0$  (obvious)

(4)  $\sum_{n=1}^{\infty} n$  diverges because  $\lim_{n \rightarrow \infty} n = \infty$

Theorem (2.7.4) Comparison Test.

Let  $(a_k), (b_k)$  be sequences w/  $0 \leq a_k \leq b_k \forall k \in \mathbb{N}$ .

Then

(i) If  $\sum_{k=1}^{\infty} b_k$  converges  $\Rightarrow \sum_{k=1}^{\infty} a_k$  converges.

(ii) If  $\sum_{k=1}^{\infty} a_k$  diverges  $\Rightarrow \sum_{k=1}^{\infty} b_k$  diverges.

Example Since  $\frac{1}{n^2 \ln n} < \frac{1}{n^2} \quad \forall n \geq 2 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2 \ln n}$

converges. ( $\sum \frac{1}{n^2}$  converges)

Proof: Both statements follow from Thm. 1.7.2 (Cauchy Criterion for Series) and the fact that if  $a_k \leq b_k \ \forall k \in \mathbb{N}$ , then

$$|a_{m+1} + a_{m+2} + \dots + a_n| \leq |b_{m+1} + b_{m+2} + \dots + b_n|. \quad \square$$

Note: In the Comparison Test, we can make the condition  $0 \leq a_k \leq b_k$  hold for all  $k \geq M$  for some  $M$ .

### • More Series & Tests :

• Geometric Series:  $\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + ar^3 + \dots$   
 If  $r=1$  ( $a \neq 0$ )  $\Rightarrow \sum_{k=0}^{\infty} a = a \sum_{k=0}^{\infty} 1$  diverges.

If  $r \neq 1$ , then:

from  $(1-r)(1+r+r^2+r^3+\dots+r^{m-1}) = 1-r^m \Rightarrow$

$$S_m = a + ar + ar^2 + \dots + ar^{m-1} = \frac{a(1-r^m)}{1-r}$$

and  $\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$  (series converges to  $\frac{a}{1-r}$ )

iff  $|r| < 1$ . (For  $|r| \geq 1$ : diverges)

Recall:  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges iff  $p > 1$  (Ex. 2.4)

To prove it, use Cauchy Condensation Test and geometric series results. (Exer. 2.7.5)

Theorem 2.7.6 Absolute Convergence Test,

If the series  $\sum_{n=1}^{\infty} |a_n|$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges as well.

Proof: Note that if  $\sum_{n=1}^{\infty} |a_n|$  converges, ④  
 then  $\forall \varepsilon > 0 \exists N \in \mathbb{N}$  s.t.  $\forall n > m \geq N \Rightarrow$   
 $|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$  (Cauchy Crit. Test)

Triangle inequality:

$$|a_{m+1} + a_{m+2} + \dots + a_n| \leq |a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$$

$\underbrace{|S_m - S_n|}_{\sum_{n=1}^{\infty} a_n \text{ converges as well by Cauchy Crit. for Series}}$

Note: The converse is false: □

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (\text{see § 2.1})$$

converges, but  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$  diverges  
 (harm. series)

by

Theorem 2.7.7 Alternating Series Test.

Let  $(a_n)$  be a sequence s.t.

(1)  $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$  and

(2)  $\lim_{n \rightarrow \infty} a_n = 0$

Then, the alternating series  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

Proof: Exer. 2.7.1

Def: If  $\sum_{n=1}^{\infty} |a_n|$  converges, then we say that  $\sum_{n=1}^{\infty} a_n$  converges absolutely. If  $\sum_{n=1}^{\infty} |a_n|$  does not converge, but  $\sum_{n=1}^{\infty} a_n$  converges, we say  $\sum_{n=1}^{\infty} a_n$  converges conditionally.

(5)

Example: Consider  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ . It converges conditionally (since it diverges), whereas

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$  converges absolutely (since  $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$  converges).

- Back to  $\sum_{n=1}^{\infty} \frac{1}{n^p}$ . It converges  $\Leftrightarrow p > 1$ . (Corollary 2.4.7)

Indeed:

$$\text{Use } \underbrace{\sum_{n=0}^{\infty} 2^n b_{2^n}}_{2^n} = \sum_{n=0}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=0}^{\infty} 2^{(1-p)n},$$

from CCT

it is a geometric series w/  $a=1$ ,  $r=2^{1-p}$   
which converges iff  $|2^{1-p}| = 2^{1-p} < 1$  iff  $1-p < 0$   
 $\Leftrightarrow p > 1$  as desired.

Other tests:

- Ratio Test (Exer. 2.7.9)

$\sum_{n=1}^{\infty} a_n$  w/  $a_n \neq 0$  converges absolutely if

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = r < 1 \quad (r > 1 \Rightarrow \text{diverges}, r = 1 \Rightarrow \text{inconclusive})$$

- Root Test

$\sum_{n=1}^{\infty} a_n$  converges absolutely if

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = r < 1 \quad (r > 1 \text{ diverges}, r = 1 \text{ inconclusive})$$

• Abel's Test (Exer. 2.7.13)

If  $\sum_{n=1}^{\infty} a_n$  converges and if  $(b_n)$  is a sequence with  $b_1 \geq b_2 \geq \dots \geq 0$ , then  $\sum_{n=1}^{\infty} a_n b_n$  converges.

Examples :

- ① Use as Divergence Criterion:

if  $\lim_{n \rightarrow \infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n$  diverges. (See Thm. 2.7.3)

For example,  $\lim_{n \rightarrow \infty} (-1)^n$  DNE  $\Rightarrow \sum_{n=1}^{\infty} (-1)^n$  diverges

$\lim_{n \rightarrow \infty} n = \infty \Rightarrow \sum_{n=1}^{\infty} n$  diverges

$\lim_{n \rightarrow \infty} 1 = 1 \Rightarrow \sum_{n=1}^{\infty} 1$  diverges.

$$\textcircled{2} \quad \sum_{n=0}^{\infty} \frac{e^{-n}}{n + \cos^2 n}$$

Note:  $\frac{e^{-n}}{n + \cos^2 n} \leq \underbrace{\frac{e^{-n}}{n}}_{\geq 0} \leq \frac{e^{-n}}{1} = e^{-n} = \frac{1}{e^n}$

$\sum_{n=0}^{\infty} \frac{1}{e^n}$  converges as a geom. series w/  $a=1$  and  $r=\frac{1}{e}<1$ .

Hence, by the Comparison Test,

$$\sum_{n=0}^{\infty} \frac{e^{-n}}{n + \cos^2 n} \text{ converges as well.}$$

$$\textcircled{3} \quad \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$$

Note  $\cos(n\pi) = (-1)^n \Rightarrow$

$$\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}} \text{ where } \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} = 0$$

and  $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}} \quad \forall n$ . Thus, the series converges

by the Alternating Series Test. (anso) (7)

Note: since  $\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$   
 w/  $p = 1/2 < 1 \Rightarrow$  diverges. That is,  $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}}$   
 converges conditionally.

$$(4) \quad \sum_{n=0}^{\infty} \frac{n!}{5^n}$$

$\lim_{n \rightarrow \infty} \left| \frac{(n+1)!}{5^{n+1}} \cdot \frac{5^n}{n!} \right| = \lim_{n \rightarrow \infty} \left| \frac{n+1}{5} \right| = \infty \Rightarrow$   
 by the Ratio test, the series diverges

$$(5) \quad \sum_{n=1}^{\infty} \frac{n^n}{3^{2n+1}}$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{3^{2n+1}}} = \lim_{n \rightarrow \infty} \frac{n}{3^{\frac{2n+1}{n}}} = \lim_{n \rightarrow \infty} \frac{n}{3^{2+1/n}} = \infty$$

So, the series diverges by the Root test.

$$(6) \quad \sum_{n=1}^{\infty} \frac{1}{2^n (n^2+n)} \text{ converges by Abel's Test: } (\sum a_n b_n \text{ w/ } \sum a_n \text{ conv, } b_n \downarrow, b_n > 0)$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ converges and } \frac{1}{n^2+n} > 0 \text{ and } \downarrow.$$

## Rearrangements of Infinite Series

In Section 2.1 we considered the following:

$$(1) \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Rearranged as:  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \dots$

We could try other rearrangements ...

What is a rearrangement?

Def: For a series  $\sum_{k=1}^{\infty} a_k$ , a series  $\sum_{k=1}^{\infty} b_k$  is called a rearrangement of  $\sum_{k=1}^{\infty} a_k$  if  $\exists$  bijection  $f: \mathbb{N} \rightarrow \mathbb{N}$  s.t.  $b_{f(k)} = a_k \forall k \in \mathbb{N}$ .  
 (It is a permuted sum w/ all the original terms included without repetitions.)

Recall: for  $\sum_n (-1)^{n+1} = S$ , we got that  $\frac{3}{2}S = S$  after a (mentioned above) rearrangement.

The issue here is that  $\sum_n (-1)^{n+1}$  converges conditionally! ( $\sum |(-1)^{n+1}| = \sum n$  diverges)

Theorem (2.7.10) If a series converges absolutely then any rearrangement of the series converges to the same limit.

Other series in §d.1:

$$\textcircled{2} \quad S = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots \quad \text{converges}$$

to the same limit as the rearranged version

$$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} + \frac{1}{64} - \frac{1}{8} + \dots \quad \text{because}$$

$$\sum_{n=0}^{\infty} \left| \left(-\frac{1}{2}\right)^n \right| = \sum_{n=0}^{\infty} \frac{1}{2^n} \text{ is a convergent geom. series.}$$

That is,  $\sum (-\frac{1}{2})^n$  converges absolutely  $\Rightarrow$  any rearrangement has the same limit.

③ Lastly, we looked at this series diverges by the Div. Criterion.

$$\lim_{n \rightarrow \infty} (-1)^n \text{ DNE.}$$

$$S = \sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 \dots$$

recall:  
2 diff. groupings:

$$0 \swarrow \quad \searrow -1$$

Proof of Thm. 2.7.10.

absolutely

Let  $\sum_{k=1}^{\infty} a_k$  be convergent to  $S$ , and let  $\sum_{k=1}^{\infty} b_k$  be a rearrangement of  $\sum_{k=1}^{\infty} a_k$ .

Let  $S_n = \sum_{k=1}^n a_k = a_1 + a_2 + \dots + a_n$  and

$t_m = \sum_{k=1}^m b_k = b_1 + b_2 + \dots + b_m$  be partial sums for  $\sum_{k=1}^n a_k$  and  $\sum_{k=1}^{\infty} b_k$ , respectively.  
We will show that  $t_m \xrightarrow[m \rightarrow \infty]{} S$ .

Pick  $\epsilon > 0$ . Since  $S_n \rightarrow S$ , then for some  $N_1$ ,  $\forall n \geq N_1 \Rightarrow |S_n - S| < \epsilon/2$ . For absolute convergence, we will use the Cauchy criterion and for the same  $\epsilon$ , find  $N_2$  s.t.  $\forall n > m \geq N_2$ ,

$$|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \frac{\epsilon}{2}.$$

$\underbrace{|S_n - S_m|}_{|\bar{S}_n - \bar{S}_m|}$  where  $(\bar{S}_k)$  is a partial sum for  $\sum |a_k|$

Now let  $N = \max \{N_1, N_2\}$ . Recall

$b_{f(k)} = a_k \quad \forall k \in N$ . We know that  $\{a_1, a_2, \dots, a_N\}$  must appear in  $\sum_{k=1}^{\infty} b_k$ , so choosing  $M = \max \{f(k), 1 \leq k \leq N\}$ , we can

(10)

guarantee that  $t_m \geq M$ ,  $|t_m - s_N|$

consists of a finite # terms with the  $\sum_{k=N+1}^{\infty} |a_k|$ . Note also,  $|t_m - s_N| < \frac{\epsilon}{2}$ .

$$\text{Then } |t_m - s| = |t_m - s_N + s_N - s| \leq |t_m - s_N| + |s_N - s| \\ = \epsilon \text{ for } m \geq M. \quad \square$$

$$\textcircled{*} \quad |t_m - s_N| \leq \underbrace{\text{sum of finite \#}}_{\text{finite \# terms}} \underbrace{< \frac{\epsilon}{2}}_{\text{triangle inequality}} \underbrace{\text{of abs. values of}}_{\text{the terms}} \underbrace{\text{by abs. convergence}}_{\text{and choice of } N_2} \underbrace{\text{and choice of } N_2}_{\text{and choice of } N_2}$$

Notice that abs. convergence is a desirable quality to have for manipulating series!  
Any rearrangement  $\rightarrow$  same limit.

If  $\sum a_k$  converges conditionally, then its rearrangements are no longer guaranteed to converge to the same limit. More than that,  $t \in \mathbb{R}$   $\exists$  a rearrangement of  $\sum a_k$  that converges to  $r$ .