

Chapter 3 : Basic Topology of \mathbb{R} .

①

§ 3.1 Cantor Set.

Recall: Georg Cantor proved that \mathbb{R} is uncountable. Let us now discuss another interesting Cantor's idea: the Cantor set.

Let $C_0 = [0, 1]$

$$C_1 = C_0 \setminus \left(\underbrace{\left(\frac{1}{3}, \frac{2}{3} \right)}_{\text{open middle third is removed}} \right) = \left[0, \frac{1}{3} \right] \cup \left[\frac{2}{3}, 1 \right]$$

$$C_2 = C_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right)$$

$$= \left(\left[0, \frac{1}{9} \right] \cup \left[\frac{2}{9}, \frac{1}{3} \right] \right) \cup \left(\left[\frac{2}{3}, \frac{7}{9} \right] \cup \left[\frac{8}{9}, 1 \right] \right)$$

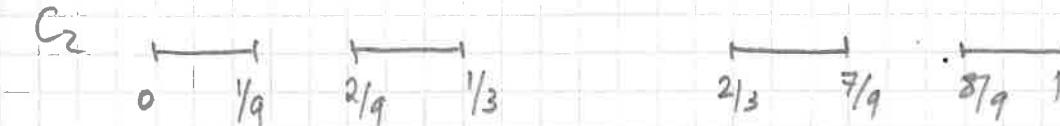
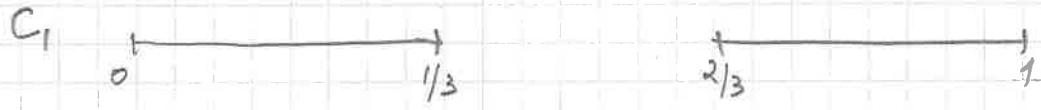
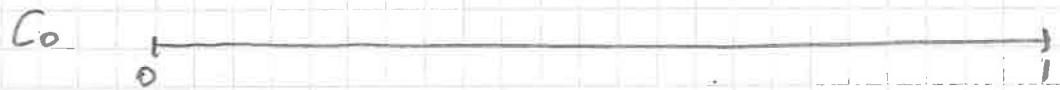
$\left(\frac{1}{9}, \frac{2}{9} \right)$ is removed $\left(\frac{7}{9}, \frac{8}{9} \right)$ is removed

Continue ... $(n = 0, 1, 2, \dots)$

C_n will consist of 2^n closed intervals of length $\frac{1}{3^n}$.

The Cantor set is

$$C = \bigcap_{n=0}^{\infty} C_n$$



Note: Cantor set is a fractal (self-similar set replicating itself at different scales)

Note $C \neq \emptyset$ since $0 \in C$ & $1 \in C$.

More than that, if y is the endpoint of a closed interval in C_n , then it is also an endpoint of an interval in C_{n+1} . Thus, $y \in C_n$ for all n . Hence, C ^{at least} contains the endpoints of all the intervals that make up sets C_n .

- Questions:
- Is C countable?
 - What else does it contain?
 - Does it contain any irrationals?

(endpts are all rationals)

If C had only endpoints $\frac{m}{3^n}$ then
 $C \subset \mathbb{Q} \Rightarrow C$ 'd be countable. (Not sure yet...)

Let's see:

$C_1 \leftarrow$ open interval of length $\frac{1}{3}$ is removed

$C_2 \leftarrow$ two open intervals of length $\frac{1}{9}$ each
 $\vdots \quad \vdots$ are removed

$C_n \leftarrow 2^{n-1}$ open intervals of length $\frac{1}{3^n}$
 are removed

We could think of "length" of C as

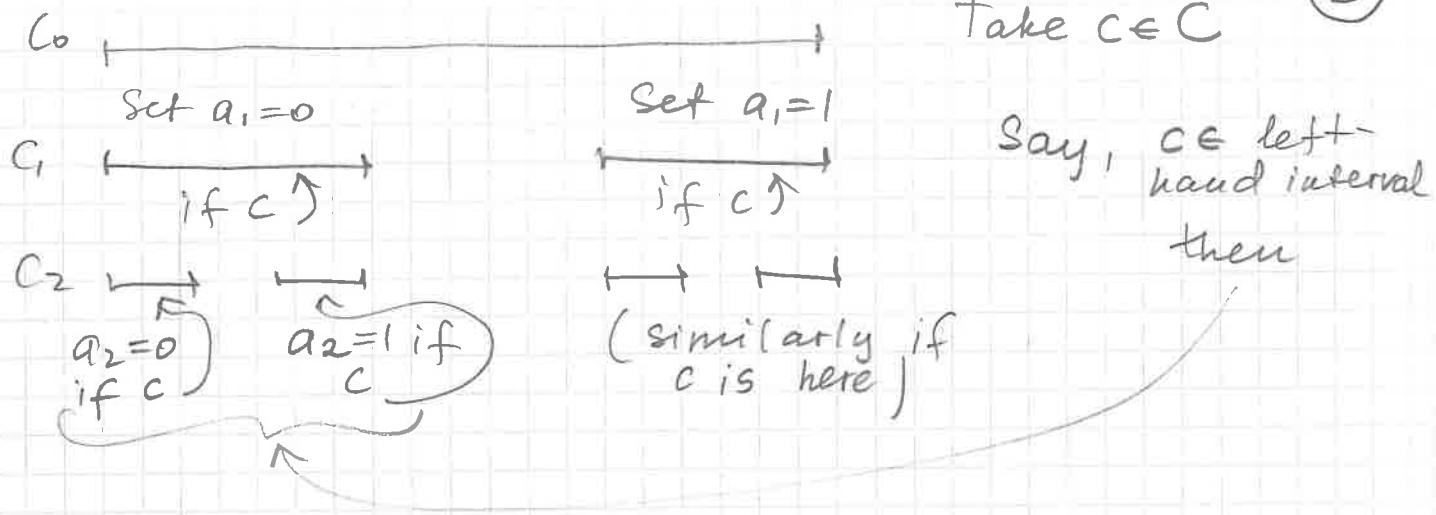
$$1 - \left(\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \dots + 2^{n-1}\left(\frac{1}{3^n}\right) + \dots \right)$$

$$= 1 - \sum_{n=0}^{\infty} \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^n = 1 - \frac{\frac{1}{3}}{1 - \frac{2}{3}} = 1 - 1 = 0.$$

"zero length"

So, C is small? (Notice C is referred to as "Cantor dust") No! In fact, C is uncountable!
 And $C \sim \mathbb{R}$.

(3)



So $\forall c \in C$, \exists a sequence (a_1, a_2, a_3, \dots) of 0's & 1's (giving directions to locate c in the set).

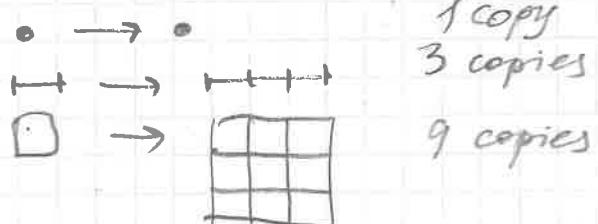
On the other hand, each such sequence corresponds to some $c \in C$. Thus, we just confirmed that \exists 1-1 correspondence between C and $S = \{(a_1, a_2, a_3, \dots) : a_n = 0 \text{ or } 1\}$.

Exer. 1.6.4 $\Rightarrow S$ is uncountable $\Rightarrow C$ is uncountable as well.

Thus, C is large (in the cardinality context) & $C \sim \mathbb{R}$. Recall, when we discussed length of C , we got 0. (basically, C measures the same size as a point). \exists another category: dimension.

	dim
point	0
line segment	1
square	2
cube	3

Magnifying by a factor of 3:





27 copies

	dim	$\times 3$	copies
pt	0	\rightarrow	$1 = 3^0$
segment	1	\rightarrow	$3 = 3^1$
square	2	\rightarrow	$9 = 3^2$
cube	3	\rightarrow	$27 = 3^3$
C	x	\rightarrow	$2 = 3^x$

why?

C?

$$C_0 = [0,1] \xrightarrow{\times 3} [0,3]$$

Deleting $\frac{1}{3}$ in the middle $\Rightarrow [0,1] \cup [2,3]$

(C_1) $\underbrace{[0,1]}_{\text{2 copies of } C_0} \cup [2,3]$

In each of these 2 intervals, we produce a copy of C_1
 \Rightarrow total 2 copies, and so on...

Magnifying C by a factor of 3 produces 2 copies of C.

$$\text{If } \dim C = x \Rightarrow 2 = 3^x \Rightarrow x = \frac{\ln 2}{\ln 3} \approx 0.63$$

C is a fractal w/ non-integer dimension!
 1975, Benoit Mandelbrot

This chapter: discussion of topology of \mathbb{R} , namely: open, closed, compact, connected sets, etc. (We can also use this material to explore C more.)