

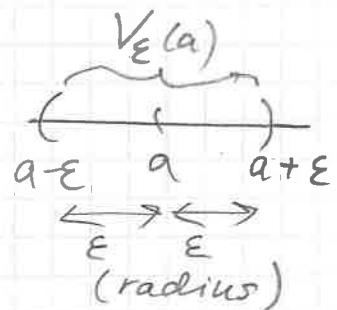
(1)

§ 3.2

Open & Closed Sets.

Recall ε -neighborhood of a :

$$V_\varepsilon(a) = \{x \in \mathbb{R} : |x-a| < \varepsilon\}$$



Def. A set $O \subset \mathbb{R}$ is open if
 $\forall a \in O \exists V_\varepsilon(a) \subset O$.

Examples : (1) \mathbb{R} itself ($\forall a \in \mathbb{R}$, pick any $V_\varepsilon(a) \subset \mathbb{R}$)
(2) \emptyset

$$(3) \forall (a, b) = \{x \in \mathbb{R} : a < x < b\}$$

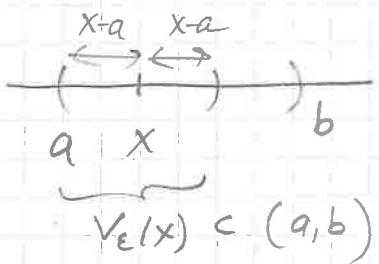
$$(4) \bigcup_i (a_i, b_i) \quad \& \quad \bigcap_{i=1}^n (a_i, b_i)$$

Theorem (3.2.3)

(1) The union of an arbitrary collection of open sets is open.

(2) The intersection of a finite collection of open sets is open.

take $\forall x \in (a, b) \Rightarrow$
if $\varepsilon = \min\{x-a, b-x\}$
 $\Rightarrow V_\varepsilon(x) \subset (a, b)$



(Infinite intersection may not be open: $\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ not open)

Proof:

(1) $\{O_i : i \in I\}$ - collection of open sets.

Let $O = \bigcup_{i \in I} O_i$. Take $\forall a \in O \Rightarrow \exists i^* \in I$ s.t.
 $a \in O_{i^*}$. Since O_{i^*} is open $\Rightarrow \exists V_\varepsilon(a) \subset O_{i^*}$

and hence $V_\varepsilon(a) \subset \bigcup_{i \in I} O_i = \emptyset$. Thus, O is open by definition. (2)

(2) Let $\{O_1, O_2, \dots, O_n\}$ be a finite collection of open sets. Taking $\forall a \in \bigcap_{i=1}^n O_i \Rightarrow a \in O_i \ \forall i$ \Rightarrow since O_i is open, $\exists V_{\varepsilon_i}(a) \subset O_i \ \forall i$. Let $\varepsilon = \min \{\varepsilon_i\}_{1 \leq i \leq n} \Rightarrow V_\varepsilon(a) \subset V_{\varepsilon_i}(a) \ \forall i \Rightarrow V_\varepsilon(a) \subset \bigcap_{i=1}^n O_i \Rightarrow$ open. □

Def: A pt. x is a limit pt of a set A if $\forall V_\varepsilon(x)$ of x contains a point of A different from x .

[Limit pts are often called "cluster points" or "accumulation points".]

$\Rightarrow \forall V_\varepsilon(x), V_\varepsilon(x) \cap A$ contains $y \neq x$ ($y \in A$)

Theorem (3.2.5)

x is a limit pt of A iff $x = \lim_{n \rightarrow \infty} a_n$ for some sequence $(a_n) \subset A$ satisfying $a_n \neq x \ \forall n \in \mathbb{N}$.

↓

to avoid $(a_n) = (x, x, x, \dots)$

Proof:

(\Rightarrow) If x is a limit pt of A , $\forall V_\varepsilon(x) \cap A$ contains a point $\neq x$. So, consider $\forall n \in \mathbb{N}$ and and $a_n \in V_{1/n}(x) \cap A$, $a_n \neq x$. $\forall \varepsilon > 0 \ \exists N$ w/ $N > 1/\varepsilon$ s.t. $\forall n \geq N > 1/\varepsilon$, $a_n \in V_{1/n}(x) \cap A$

(Archimedean prop.)

with $|a_n - x| < \frac{1}{n} \leq \frac{1}{N} < \epsilon$. That is, (3)
 $\lim_{n \rightarrow \infty} a_n = x$ for the sequence (a_n) .

(\Leftarrow) Let $(a_n) \subset A$, $a_n \neq x$, and $a_n \xrightarrow{n \rightarrow \infty} x$.

Pick $\epsilon > 0$, then $\exists N$ s.t. $\forall n \geq N \Rightarrow |a_n - x| < \epsilon$. That is, $\forall \epsilon > 0 \exists a_N \in V_\epsilon(x) \Rightarrow x$ is a limit point of A . □

Examples : 1) Points 0 & 1 are both limit pts

$[0,1]$ of the interval $(0,1)$. $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, $\lim_{n \rightarrow \infty} 1 - \frac{1}{n} = 1$
 $\overbrace{}^0$ $\overbrace{}^1$
 $\underbrace{}_{(n=1)}$

set of all limit pts of $(0,1)$ 2) Set $\mathbb{Z} \subset \mathbb{R}$ has no limit pts

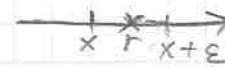
set of all limit pts of \mathbb{Q} 3) $A = (0,2) \cup \{3\}$

- (i) 1 is a limit pt of A & $1 \in A$
- (ii) 0 is a limit pt of A & $0 \notin A$
- (iii) 3 is not a limit pt of A , $3 \in A$.
- (iv) -1 is not a limit pt of A , $-1 \notin A$.

4) The set of all limit pts of (a,b) is $[a,b]$.

5) The set of limit pts of \mathbb{Q} is \mathbb{R} .

$(\forall x \in \mathbb{R}, \forall \epsilon > 0 \exists r \in \mathbb{Q} \text{ w/ } x < r < x + \epsilon, r \neq x \text{ and } r \in V_\epsilon(x) \cap \mathbb{Q})$



Def: A pt. $a \in A$ is an isolated pt of A if it is not a limit pt of A . $\forall \epsilon > 0$ is isolated;
 3 is an isolated pt for $A = (0,2) \cup \{3\}$

Notice: an isolated pt of A is in A, while a limit pt. of A may not belong to A .

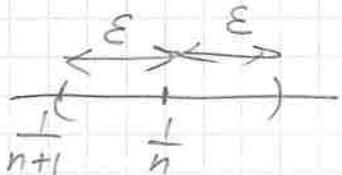
[Ex: $1 \notin (0,1)$, 1 is a limit pt. of $(0,1)$]

Examples : 0) \mathbb{Q} in \mathbb{R} is isolated.

(4)

1) \mathbb{Q} has no isolated pts (every $x \in \mathbb{Q} \subset \mathbb{R}$ a limit pt of \mathbb{Q} . Recall: \mathbb{R} is a set of limit pts for \mathbb{Q})

2) Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Claim: $\forall a \in A$ is isolated. Note: $\forall n \in \mathbb{N}, \frac{1}{n} \in A$. Let $\epsilon = \frac{1}{n} - \frac{1}{n+1} > 0$. Then $V_\epsilon(\frac{1}{n}) \cap A = \{\frac{1}{n}\} \neq \emptyset, b \in A, b \neq \frac{1}{n}$.



Thus, $\frac{1}{n}$ is an isolated pt. $\forall n \in \mathbb{N}$

However, A has one limit pt, $0 \notin A$ ($\lim_{n \rightarrow \infty} \frac{1}{n} = 0$)

Def: A set $F \subset \mathbb{R}$ is closed if it contains its limit points.

Examples :

1) $[a, b]$ is a closed set (the limit pts $a, b \in [a, b]$)

2) $A = \{\frac{1}{n}, n \in \mathbb{N}\} \Rightarrow A \cup \{0\}$ is a closed set.

3) \mathbb{Q} is not closed. \mathbb{Q} is not open either!

($\forall x \in \mathbb{Q}, \forall \epsilon > 0 : V_\epsilon(x)$ contains irrationals \Rightarrow

$\nexists V_\epsilon(x) \subset \mathbb{Q}$ only $\Rightarrow \mathbb{Q}$ is not open. Not closed:

$\forall i \in \mathbb{Z}, \forall \epsilon > 0 : V_\epsilon(i)$ contains rationals

$\Rightarrow \forall i \in \mathbb{Z}$ is a limit pt for \mathbb{Q} , but $i \notin \mathbb{Q}$)

4) \mathbb{Z} is closed (and not open!)

5) $\{x \in \mathbb{R} : x > 0\}$ or $(0, \infty)$ is an open set (not closed).

6) $(0, 1]$ is not open and not closed.

7) \mathbb{R} & \emptyset are both open & closed.

(5)

Theorem 3.2.8) A set $F \subseteq \mathbb{R}$ is closed iff
 & Cauchy sequence in F has a limit that is also an element of F .

Proof:

(\Rightarrow) Suppose $F \subseteq \mathbb{R}$ is closed. Let $(x_n) \subset F$ be a Cauchy sequence. $x_n \xrightarrow[n \rightarrow \infty]{\forall n \in \mathbb{N}} x$ for some $x \in \mathbb{R}$ (all Cauchy sequences converge). If $x = x_n$ for some n , then we are done. If $x \neq x_n \forall n$, then by Thm. 3.2.5, we know that x is a limit pt of F , and since F is closed, $x \in F$.

(\Leftarrow) Let F be a set s.t. & Cauchy seq. in F has a limit $\in F$. Let x be a limit pt. for F . By Thm 3.2.5, \exists a seq. (x_n) , $x_n \in F \forall n$, $x_n \neq x \forall n$ & $x_n \xrightarrow[n \rightarrow \infty]{} x$. Since (x_n) converges $\Rightarrow (x_n)$ is Cauchy & so $x \in F$. Since our choice of x was arbitrary $\Rightarrow F$ contains all of its limit pts $\Rightarrow F$ is closed. \square

Theorem 3.2.10) Density of \mathbb{Q} in \mathbb{R} ← revisited.

$\forall y \in \mathbb{R} \exists$ a sequence of rational numbers that converges to y .

(Thm. (1.4.3): $\forall a, b \in \mathbb{R}, a < b, \exists r \in \mathbb{Q}$ w/ $a < r < b$)

Recall all limit pts of \mathbb{Q} are all of \mathbb{R} .

$\forall y \in \mathbb{R} \exists \varepsilon > 0$ $\forall \varepsilon > 0$ $V_\varepsilon(y) = (y - \varepsilon, y + \varepsilon)$ contains $r \in \mathbb{Q}$, $r \neq y$

so y is a limit pt of $\mathbb{Q} \Rightarrow \exists$ a seq. in \mathbb{Q} that converges to y .

Def: For a set $A \subset \mathbb{R}$, let L be the set of all limit pts of A . The closure of A is $\bar{A} = A \cup L$. (\bar{A} is always a closed set.) (6)

Examples :

- 1) Recall $A = \{\frac{1}{n} : n \in \mathbb{N}\} \Rightarrow \bar{A} = A \cup \{0\}$
- 2) $\bar{\mathbb{Q}} = \mathbb{R}$
- 3) If $A = (a, b) \Rightarrow \bar{A} = [a, b]$
- 4) If $A = [a, b] \Rightarrow \bar{A} = A$.

Theorem (3.2.12) $\forall A \subset \mathbb{R}$, \bar{A} is a closed set and is the smallest closed set containing A .

Proof : Clearly, \bar{A} contains the limit pts of A . Question: will $\bar{A} = A \cup L$ potentially produce more limit pts of \bar{A} ? The answer is no. (Exercise 3.2.7)

Next, \forall closed set containing A must contain L as well. Hence, $\bar{A} = A \cup L$ is the smallest set containing A . □

Important: If set is not open $\not\Rightarrow$ it must be closed!

- \mathbb{R} & \emptyset are both open & closed the only ones w/ this property!
- Intervals $(a, b] = \{x \in \mathbb{R}, a < x \leq b\}$ are neither open nor closed

• However, there is an important relationship between open & closed sets.

Recall: $A^c = \{x \in \mathbb{R}, x \notin A\}$ is the complement of $A \subset \mathbb{R}$.

Theorem (3.2.13) (1) A set O is open if and only if O^c is closed. (2) Likewise, a set F is closed if and only if F^c is open.

Proof: (1) \Rightarrow Let O be open in \mathbb{R} . Consider any limit pt x of O^c . Then $\forall V_\epsilon(x)$ contains some pt of O^c . Note $x \notin O$ because if $x \in O$ $\Rightarrow \exists V_\epsilon(x) \subset O$ (O is open) which is not possible. Thus, $x \in O^c$. Since x was arbitrary, then any limit pt of O^c is in $O^c \Rightarrow O^c$ is closed.

\Leftarrow Now let O^c be closed. Take any $x \in O$. Since O^c is closed \Rightarrow x cannot be a limit pt of $O^c \Rightarrow \exists V_\epsilon(x) \cap O^c = \emptyset$.

This means that $V_\epsilon(x) \subset O \Rightarrow O$ is open, as $\forall x \in O \exists V_\epsilon(x) \subset O$.

Part (2): follows immediately from (1) using the fact that $(A^c)^c = A \quad \forall A \subset \mathbb{R}$. \square

Theorem (3.2.14) (Compare to Thm. 3.2.3)

(i) The union of a finite collection of closed sets is closed.

(ii) The intersection of an arbitrary collection of closed sets is closed.

Proof: see text.

Examples :

(for $a \in \mathbb{R}$)

(8)

(1) The intervals $(-\infty, a)$, (a, ∞) are both open ($\forall x$ is included with a neighborhood in the interval)

(2) The intervals $[a, \infty)$ and $(-\infty, a]$ are both closed because their complements

$([a, \infty))^c = (-\infty, a)$ and $((-\infty, a])^c = (a, \infty)$ are open.

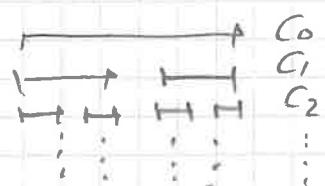
(3) $I = (0, 1]$ is neither open nor closed:

I is not open because it does not contain any $V_\epsilon(1)$. $I^c = (-\infty, 0] \cup (1, \infty)$ isn't open either, I^c doesn't contain $\forall V_\epsilon(0)$. Thus, I is not closed.

(4) Recall \mathbb{Q} is neither open nor closed:

\mathbb{Q} is not open since $\forall V_\epsilon(r)$, $r \in \mathbb{Q}$, contains irrational #'s, and $\mathbb{Q}^c = \mathbb{I}$ isn't open since $\forall V_\epsilon(q)$, $q \in \mathbb{I}$, contains rational #'s $\Rightarrow \mathbb{Q}$ is not closed either.

(5) The Cantor set $C = \bigcap_{n=0}^{\infty} C_n$



by Thm 3.2.14

Note : $\forall n$, C_n is closed as a union of finite collection of closed sets
 $\Rightarrow C$ is closed as the intersection of closed sets.

removed open sets

$$\left\{ \begin{array}{l} I_1 = \left(\frac{1}{3}, \frac{2}{3}\right) \\ I_2 = \left(\frac{1}{3^2}, \frac{2}{3^2}\right) \cup \left(\frac{7}{3^2}, \frac{8}{3^2}\right) \\ \vdots \end{array} \right.$$

(Recall: C is also uncountable, of zero length)