

### § 3.3. Compact Sets.

Finite sets are nice: contain their sup/inf  
 bounded      (max/min)  
 closed

Def: Compactness. Compact sets too!

A set  $K \subset \mathbb{R}$  is compact if every sequence in  $K$  has a subsequence that converges to a limit that is also in  $K$ .

Example: any closed interval. Why?

If a sequence  $(a_n)$  is in  $[x,y]$ , i.e.  $x \leq a_n \leq y$ , then it is bounded, and by BW theorem, it contains a convergent subsequence. Since  $[x,y]$  is closed then the limit of such a subsequence is in  $[x,y]$ .

Def: Bounded Sets.

A set  $A \subset \mathbb{R}$  is bounded if  $\exists M > 0$  s.t.  $|a| \leq M \forall a \in A$ .

Theorem (3.3.4) Compactness in  $\mathbb{R}$ .

A set  $K \subset \mathbb{R}$  is compact iff  $K$  is closed and bdd.

Proof:

$(\Rightarrow)$  let  $K$  be compact. Assume, by contrad., that  $K$  is not bdd. Then  $\exists x_1 \in K$  w/  $|x_1| > 1$ .

Likewise,  $\exists x_2 \in K$  w/  $|x_2| > 2$ , ...,  $\exists x_n \in K$  w/  $|x_n| > n$ , and so on.  $\rightarrow (\forall M > 0 \exists a \text{ s.t. } |a| > M)$

So, we have  $(x_n) \subset K$  w/  $|x_n| > n \ \forall n$ . (2)

Recall:  $K$  is compact  $\Rightarrow \exists$  a convergent subsequence of  $(x_n)$ ,  $(x_{n_k}) \xrightarrow[n \rightarrow \infty]{\text{a limit pt.}} x \in K$ . Note that  $(x_{n_k})$  must also satisfy  $|x_{n_k}| > n_k \Rightarrow (x_{n_k})$  is unbdd. We have a contradiction, since convergent sequences must be bdd. Thus,  $K$  cannot be unbdd.

Next, show  $K$  is closed, i.e.  $K$  contains its limit pts. Let  $x = \lim_{n \rightarrow \infty} x_n$ , where  $(x_n) \subset K$ .

Since  $K$  is compact  $\Rightarrow \exists (x_{n_k}) \subset K$ , subseq. of  $(x_n)$  that converges to a limit  $\in K$ , but it must be  $x$ . Thus,  $x \in K$ . So,  $K$  is closed.

( $\Leftarrow$ ) Let  $K$  be bdd & closed. Then since  $\forall$  seq. is  $K$  is bdd  $\Rightarrow$  by BW thm, must have a convergent subseq. Since  $K$  is closed, then the limit of the subseq. is in  $K$ . Therefore,  $K$  is compact by def.

□

Example: Compact sets can be "more interesting" than closed intervals. Take, for example, the Cantor set.  $C = \bigcap_{n=0}^{\infty} C_n \subset [0, 1]$  by construction  $\Rightarrow$  bdd. It is also closed (as an intersection of closed sets by Thm. 3.2/4). Thus,  $C$  is compact by Thm. 3.3.4

Note: Compact sets can be thought of as generalizations of closed intervals.

Theorem (3.3.5)

(Compare to NIP, Thm. 1.4.1) (3)

If  $K_1 \supset K_2 \supset K_3 \supset \dots$  is a nested sequence of nonempty compact sets  $\Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$ .

Proof: see p. 97 of the text.

Open Covers

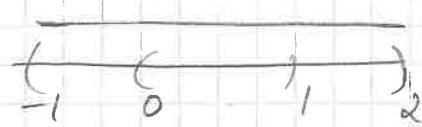
(Recall:  $K$  is compact iff  $K$  is closed & bdd.)

Another important characterization of compactness  $\rightarrow$  open covers & finite subcovers

Can be infinite

Def: Let  $A \subset \mathbb{R}$ . An open cover for  $A$  is a collection of open sets  $\{O_\lambda : \lambda \in \Lambda\}$  w/  
 $\overrightarrow{A \subset \bigcup_n O_\lambda}$ . Given an open cover for  $A$ ,  
a finite subcover is a finite subcollection of  
 $\{O_\lambda : \lambda \in \Lambda\}$  whose union still manages to  
completely contain  $A$ .

Examples:

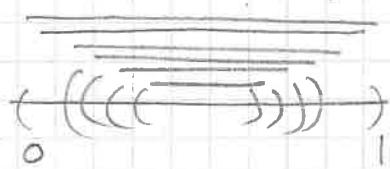


(a)  $\{(-1, 2)\}$  covers  $(0, 1)$

(b)  $V_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ ,  $n \geq 3$

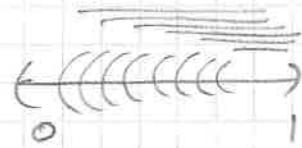
$\{V_n\}_{n=3}^{\infty}$  covers  $(0, 1)$

too.



(c) From text (Ex. 3.3.7).

Let  $O_x = \left(\frac{x}{2}, 1\right)$   $\forall x \in (0, 1)$



then  $\{O_x, x \in (0, 1)\}$  forms an open cover for  $(0, 1)$

What about a finite subcover?

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In (a) : a finite subcover is  $(-1, 2)$  itself.

In (b) : impossible to find a finite subcover!

If we choose some  $N$  and take

$$\{\underline{V_n}\}_{n=3}^N = \{V_3, V_4, \dots, V_N\} \text{ then}$$

$\downarrow$        $\downarrow$        $\downarrow$

$\circ$        $\underbrace{\hspace{1cm}}$        $\circ$

$V_N$

$\forall y \in \mathbb{R} \text{ w/ } 0 < y < \frac{1}{N} \text{ or}$   
 $1 - \frac{1}{N} < y < 1 \text{ is not in } \bigcup_{n=3}^N V_n$

In (c) : impossible either. If  $\{O_{x_1}, O_{x_2}, \dots, O_{x_n}\}$  is a finite collection from  $\{O_x, x \in (0, 1)\}$ , then

$$\text{if } \bar{x} = \min \{x_1, \dots, x_n\} \text{ then } \left(\frac{\bar{x}}{2}, 1\right) \text{ } \forall y \in \mathbb{R}, 0 < y \leq \frac{\bar{x}}{2},$$

$y \notin \bigcup_{i=1}^n O_{x_i}.$

$\downarrow$        $\overbrace{\hspace{1cm}}$

$\circ$        $\circ$        $\overbrace{\hspace{1cm}}$

Note :  $(0, 1)$  is not compact. (it is bdd, but not closed)

$[0, 1]$  is compact  $\Rightarrow$  we can find both open cover & finite subcover.

For example : fix  $\varepsilon > 0$ . Let  $O_0 = (-\varepsilon, \varepsilon)$ ,

$O_1 = (1-\varepsilon, 1+\varepsilon)$  and  $O_x = \left(\frac{x}{2}, 1\right), \forall x \in (0, 1)$

$$\left( \begin{array}{c} \cancel{\varepsilon} \\ \overset{0}{\underset{O_0}{\underbrace{\hspace{1cm}}} \quad \overset{1}{\underset{O_1}{\underbrace{\hspace{1cm}}}} \end{array} \right)$$

$\leftarrow \quad \rightarrow$   
 to cover  $0 \neq 1$

$$\frac{\bar{x}}{2} < \varepsilon$$

$$\begin{array}{ccccccc} & & O_0 & & & O_1 & \\ & \cancel{-\varepsilon} & \cancel{\varepsilon} & & \cancel{1-\varepsilon} & \cancel{1+\varepsilon} & \\ & \overset{0}{\underset{\frac{x}{2}}{\underbrace{\hspace{1cm}}} & O_x & \overset{1}{\underset{1}{\underbrace{\hspace{1cm}}}} & & & \end{array}$$

Thus, collection

$$\{O_0, O_1, O_x, x \in (0, 1)\}$$

covers  $[0, 1]$ .

Finite subcover? Yes!

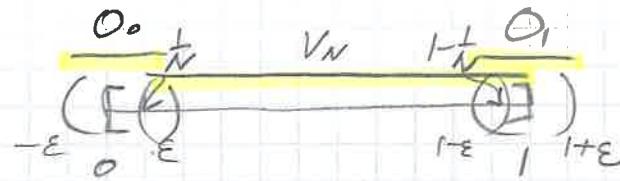
$$\{O_0, O_{\frac{x}{2}}, O_1\} \text{ where}$$

For (b) : choose  $O_0, O_1 \& V_n (n \geq 3) \Rightarrow$  we can choose

a finite subcover  
w/  $O_0, O_1$  &

$$V_N \text{ s.t. } \frac{1}{N} < \epsilon$$

$$\text{and } 1 - \frac{1}{N} > 1 - \epsilon$$



(5)

Theorem (3.3.8) Heine-Borel Theorem.

Let  $K \subset \mathbb{R}$ . The following statements are equivalent:  $\rightarrow$  recall sequential def.

- (1)  $K$  is compact  
(2)  $K$  is closed and bdd. ] Thm (3.3.4)

topolo-  $\leftarrow$  (3) Every open cover for  $K$  has a finite  
gical  
def. of  
compact  
sets  
subcover.

(any of these statements implies the two others.)

(See proof in the text)

More examples:

- (1)  $\mathbb{Z}$  is not compact, since  $\mathbb{Z}$  is not bdd.  
(recall it is closed though)

$\dots \overline{-2} \overline{-1} \overline{0} \overline{1} \overline{2} \dots$   $\leftarrow$  open cover, but  
 $\nexists$  a finite subcover

- (2)  $U_n = (-n, n)$  w/  $n \in \mathbb{N}$  ( $\mathbb{R}$  both open & closed)

The collection  $\{U_n\}_{n \in \mathbb{N}}$  is a nested open cover of  $\mathbb{R}$ .

$\dots \overline{-2} \overline{-1} \overline{0} \overline{1} \overline{2} \dots \mathbb{R}$   $\Rightarrow$  no finite subcover  
 $\mathbb{R}$  is not bdd, unctble.  
 $\Rightarrow$  not compact.

- (3)  $\mathbb{Q}$  is neither closed, nor bdd  $\Rightarrow$  not compact

- (4)  $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$  is not compact, because it is not closed:  
 $\frac{1}{n} \xrightarrow{n \rightarrow \infty} 0$ , but  $0 \notin A$ .

## § 3.4 Perfect Sets & Connected Sets. ①

Def: A set  $P \subset \mathbb{R}$  is perfect if it is closed & contains no isolated pts.

Examples : 1) closed intervals ( $\forall p \in [a,b]$  is a limit pt)  $\cup_{n=0}^{\infty} C_n$   
2) the Cantor set :  $C = \bigcap_{n=0}^{\infty} C_n$   
( $C_n$  is a finite union of closed intervals)

$C$  is closed; and one can show that  $\forall x \in C$ ,  $x$  is not isolated, that is,  $x$  is a limit pt.

Recall:  $C$ , at least, contains the endpts of the intervals that make up  $C$ . We use them to construct  $(x_n)_{\substack{n \rightarrow \infty}} \rightarrow x$ ,  $(x_n) \subset C$ . (Exer. 3.4.3)

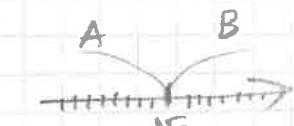
### Theorem (3.4.3)

$C$  &  $[a,b]$  are  
uncountable!

A nonempty perfect set is uncountable.

Def: Two sets  $A, B \subset \mathbb{R}$ ,  $A \neq \emptyset$ ,  $B \neq \emptyset$  are separated if  $\overline{A} \cap B = \emptyset$  &  $A \cap \overline{B} = \emptyset$ .

A set  $E \subset \mathbb{R}$  is disconnected if it can be written as  $E = A \cup B$ ; where  $A$  &  $B$  are separated sets,  $A \neq \emptyset$ ,  $B \neq \emptyset$ . A set that is not disconnected is called a connected set.

Examples : 1)  $(1, 2) \cup (2, 5)$  is disconnected  
2) Any interval  $(a, b)$  or  $[a, b]$  or  $(a, b]$  or  $[a, b)$  is a connected set.  
3)  $\mathbb{Q}$  is disconnected:  $\mathbb{Q} = A \cup B$  where  $A = \mathbb{Q} \cap (-\infty, \sqrt{2})$  and  $B = \mathbb{Q} \cap (\sqrt{2}, \infty)$ .   
4) The Cantor set is disconnected.