

# Chapter 4 : Functional Limits and Continuity.

(1)

## §§ 4.1, 4.2. Functional Limits & Examples.

Discussion about continuity of a function.

Recall:  $f(x)$  is continuous at  $x=c$  if

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \left( \begin{array}{l} \text{if we are "near } c", \\ f(x) \text{ will be "near } f(c)" \end{array} \right)$$

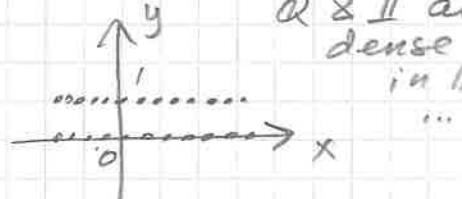
need to define

Consider Dirichlet's function:

Recall:

$\mathbb{Q}$  &  $\mathbb{I}$  are dense in  $\mathbb{R}$

$$g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$



How would we consider

$\lim_{x \rightarrow \frac{1}{2}} g(x)$ ? We could choose a seq.  $(x_n) \rightarrow \frac{1}{2}$ , but if  $x_n \in \mathbb{Q} \Rightarrow g(x_n) = 1 \Rightarrow \lim_{n \rightarrow \infty} g(x_n) = 1$ , otherwise  $(x_n \notin \mathbb{Q}) \Rightarrow g(x_n) = 0 \Rightarrow \lim_{n \rightarrow \infty} g(x_n) = 0$ .

Based on above: we need the value of limit be independent of how  $x \rightarrow c$ . In the example,

we can say that  $\lim g(x)$  does not exist (DNE)

Also,  $g(x)$  is not continuous at  $x = \frac{1}{2}$ , in fact, it is discontinuous at any  $x \in \mathbb{R}$ . We say

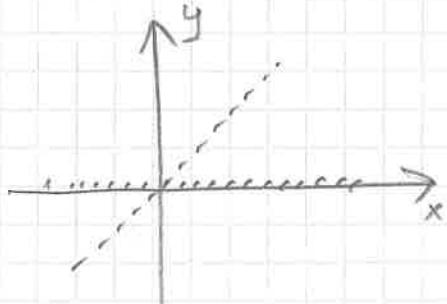
$g$  is nonwhere-continuous function on  $\mathbb{R}$ .

Another example:

$$h(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

"fixed g"

Modified Dirichlet's func.



If  $c \neq 0$  and  $x_n \rightarrow c$  w/  $x_n \in \mathbb{Q}$ ,  $y_n \rightarrow c$  w/  $y_n \notin \mathbb{Q}$ , then  $\lim_{n \rightarrow \infty} h(x_n) = c$  and  $\lim_{n \rightarrow \infty} h(y_n) = 0 \Rightarrow$  no limit as  $x \rightarrow c (\neq 0)$  (2)

If  $c=0$  then  $\lim_{n \rightarrow \infty} h(x_n) = \lim_{n \rightarrow \infty} h(y_n) = 0 = h(0) \Rightarrow h$  is cont. at 0

In fact, no matter what  $z_n$  is, but if  $z_n \rightarrow 0$ , then  $h(z_n) \rightarrow 0$ .

Observation:  $\lim_{x \rightarrow c} h(x) = L \Rightarrow h(z_n) \rightarrow L$   
 $\forall z_n \rightarrow c$   $n \rightarrow \infty$

One more interesting example:

Thomae's function (1875)

$$t(x) = \begin{cases} 1, & x=0 \\ 1/n, & x=m/n \in \mathbb{Q} \setminus \{0\} \text{ in the lowest terms; } n>0 \\ 0, & x \notin \mathbb{Q} \end{cases}$$

$t(x)$  is not cont.  $\forall x \in \mathbb{Q}$ , but it is  $\forall x \notin \mathbb{Q}$ !

if  $c \in \mathbb{Q}$ ,  $t(c) > 0$

take  $y_n \in \mathbb{Q}$ ,  $y_n \rightarrow c$

$$\Rightarrow \lim_{n \rightarrow \infty} t(y_n) = 0 \neq t(c)$$

if  $c \notin \mathbb{Q}$ , say  $c = \sqrt{2}$ :

take  $x_n \in \mathbb{Q}$ ,  $x_n \rightarrow \sqrt{2}$

$$\text{then } t(x_n) \rightarrow t(\sqrt{2}) = 0$$

(for  $y_n \notin \mathbb{Q}$ ,  $\lim t(y_n) = 0 = t(\sqrt{2})$ )

Next, we define limits for functions, discuss their properties, and consider examples.

Def. (Functional Limit  $\leftarrow$  " $\epsilon$ - $\delta$ " def.)

Let  $f: A \rightarrow \mathbb{R}$ , and let  $c \in A$ . We say that  $\lim_{x \rightarrow c} f(x) = L$  if  $\forall \epsilon > 0$  limit pt  
domain off.

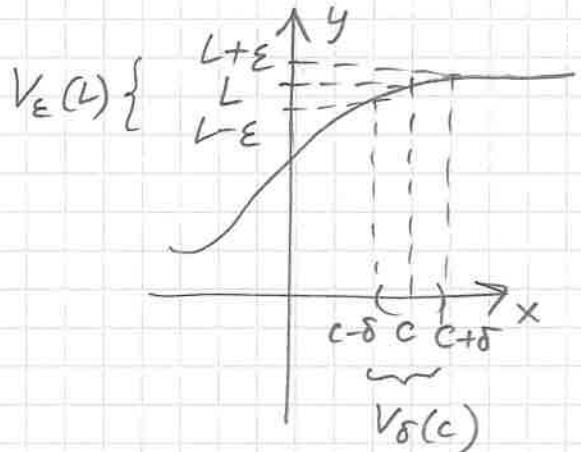
$$\exists \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon. \quad (x \in A)$$

$$\text{Also: } |f(x) - L| < \varepsilon \Leftrightarrow \quad (3)$$

$$L - \varepsilon < f(x) < L + \varepsilon, \text{ i.e., } f(x) \in V_\varepsilon(L)$$

$$\text{whereas } |x - c| < \delta \Leftrightarrow x \in V_\delta(c)$$

Restriction  $0 < |x - c|$  means  $x \neq c$



$\left\{ \begin{array}{l} \text{If } f(x) \xrightarrow[x \rightarrow c]{} L, \text{ we can get} \\ V_\varepsilon(L) \text{ as close as we like to } L \\ (\text{arbitrarily close}) \end{array} \right.$

$\left\{ \begin{array}{l} \text{provided that we are} \\ V_\delta(c) \text{ sufficiently close to } c \end{array} \right.$

Def: (Topological Version)

Let  $c$  be a limit pt of  $A \subset \mathbb{R}$ , domain of  $f$ .

We say that  $\lim_{x \rightarrow c} f(x) = L$  if  $\forall V_\varepsilon(L)$ ,

$\exists V_\delta(c)$  s.t.  $\forall x \in V_\delta(c), x \neq c \text{ and } x \in A \Rightarrow f(x) \in V_\varepsilon(L)$ .

Examples: (1) Prove that  $\lim_{x \rightarrow 2} (3x+1) = 7$ .

Let  $\varepsilon > 0$ . We want  $|3x+1 - 7| < \varepsilon$  whenever  $0 < |x-2| < \delta$  for some  $\delta = \delta(\varepsilon)$ .

So,  $|3x+1 - 7| = |3x-6| = 3|x-2| < \varepsilon \Leftrightarrow |x-2| < \frac{\varepsilon}{3}$ . Thus, if we want

$\forall \varepsilon > 0$  we choose  $\delta = \frac{\varepsilon}{3}$ , then  $\forall x, 0 < |x-2| < \frac{\varepsilon}{3}$ ,

we have  $|3x+1 - 7| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ . by def.,

$\lim_{x \rightarrow 2} (3x+1) = 7$

(2) Prove that  $\lim_{x \rightarrow 2} g(x) = 4$ , where  $g(x) = x^2$ . (4)

Pick  $\epsilon > 0$ . We want  $|x^2 - 4| < \epsilon$ .  $\delta$ ?

Note  $|x^2 - 4| = |x-2| \underbrace{|x+2|}_{\text{bound?}}$

If we agree that  $V_\delta(2)$  is no bigger than  $\delta = 1 \Rightarrow |x+2| \leq |3+2| = 5 \quad \forall x \in V_\delta(2)$ .

$$\frac{1}{|x-2|} \quad V_\delta(2) \subset (1, 3) \\ 1 < x < 3 \\ x \in (1, 3)$$

Now we want  $|x-2||x+2| \leq |x-2| \cdot 5 < \epsilon$ ,

if  $|x-2| < \min\{1, \frac{\epsilon}{5}\}$

then  $\forall \epsilon > 0 \exists \delta = \min\{1, \frac{\epsilon}{5}\}$  s.t.

$\forall x, 0 < |x-2| < \delta \Rightarrow |x^2 - 4| < \frac{\epsilon}{5} \cdot 5 = \epsilon$ .

Thus,  $\lim_{x \rightarrow 2} g(x) = 4$ .

### Theorem (4.2.3) (The Sequential Criterion)

Let  $f: A \rightarrow \mathbb{R}$  and  $c$  be a limit pt. of  $A$ . Then the following statements are equivalent:

$$(1) \lim_{x \rightarrow c} f(x) = L$$

(2)  $\forall (x_n) \subset A, x_n \neq c$  and  $\lim_{n \rightarrow \infty} x_n = c$ , it follows that  $f(x_n) \xrightarrow{n \rightarrow \infty} L$  (<sup>image seq.</sup> converges to  $L$ )

Proof : (1)  $\Rightarrow$  (2)

Let  $\lim_{x \rightarrow c} f(x) = L$  and let  $(x_n) \subset A, x_n \neq c$

w/  $x_n \xrightarrow{n \rightarrow \infty} c$ . We will show that  $f(x_n) \xrightarrow{n \rightarrow \infty} L$

Let  $\epsilon > 0$ . By top. def,  $\exists V_\delta(c)$  s.t. (5)

$\forall x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(L)$ . Recall  $x_n \xrightarrow{n \rightarrow \infty} c$   
 $\Rightarrow \exists N, \forall n \geq N, x_n \in V_\delta(c) \Rightarrow f(x_n) \in V_\epsilon(L)$   
 (for this  $\delta$ )

That is,  $\forall \epsilon > 0 \exists N$  s.t.  $\forall n \geq N |f(x_n) - L| < \epsilon$ .  
 $(\Leftrightarrow \delta \gg)$

(2)  $\Rightarrow$  (1).

Let (2) be true. Assume  $\lim_{x \rightarrow c} f(x) \neq L$ , i.e.

$\exists \bar{\epsilon} > 0$  s.t.  $\forall \delta > 0, \exists x \in V_\delta(c), x \neq c$

for which  $f(x) \notin V_{\bar{\epsilon}}(L)$ . Let  $\delta_n = \frac{1}{n}$ . Then

$\forall n (\delta_n = \frac{1}{n})$  we can find some  $x_n \in V_{\delta_n}(c)$

w/  $x_n \neq c$  s.t.  $f(x_n) \notin V_{\bar{\epsilon}}(L) \Rightarrow$  a contradiction,

since  $\forall x_n \rightarrow c, x_n \neq c \quad \underbrace{f(x_n)}_{n \rightarrow \infty} \rightarrow L$ . Thus,

(1) must be true.

$\forall V_\epsilon(L) \exists N$  s.t.  
 $\forall n \geq N f(x_n) \in V_\epsilon(L)$

□

Corollary (4.24) Algebraic Limit Theorem  
 for Functional Limits

Let  $f: A \rightarrow \mathbb{R}$  &  $g: A \rightarrow \mathbb{R}$ , and let

$\lim_{x \rightarrow c} f(x) = L$  &  $\lim_{x \rightarrow c} g(x) = M$ , where  $c$  is  
 a limit pt of  $A$ . Then

$$(1) \lim_{x \rightarrow c} kf(x) = kL \quad \forall k \in \mathbb{R}$$

$$(2) \lim_{x \rightarrow c} [f(x) + g(x)] = L + M$$

$$(3) \lim_{x \rightarrow c} [f(x)g(x)] = LM$$

$$(4) \lim_{x \rightarrow c} \left[ \frac{f(x)}{g(x)} \right] = \frac{L}{M}, M \neq 0.$$

HW...

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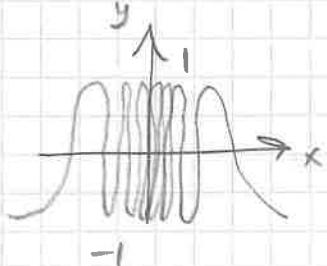
Corollary (4.2.5)

## Divergence Criterion.

Let  $f: A \rightarrow \mathbb{R}$ , and  $c$  be a limit pt of  $A$ . Then  $\lim_{x \rightarrow c} f(x)$  DNE, if  $\exists$  sequences  $(x_n) \& (y_n)$  in  $A$ ,  $x_n \neq c, y_n \neq c$  w/  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = c$ , but  $\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n)$

Example:

$$\lim_{x \rightarrow 0} \sin \frac{1}{x} \text{ DNE.}$$



$$\text{Take } x_n = \frac{1}{2\pi n} \& y_n = \frac{1}{2\pi n + \frac{\pi}{2}}$$

then

$$\sin\left(\frac{1}{x_n}\right) = \sin(2\pi n) = 0 \quad \forall n$$

whereas  $\sin\left(\frac{1}{y_n}\right) = \sin(2\pi n + \frac{\pi}{2}) = \cos(2\pi n) = 1 \quad \forall n$ . Thus,  $\lim_{n \rightarrow \infty} \sin\left(\frac{1}{x_n}\right) \neq \lim_{n \rightarrow \infty} \cos\left(\frac{1}{y_n}\right)$  while  $x_n \rightarrow 0, y_n \rightarrow 0$ . By Cor. 4.2.5, the limit DNE.

More Examples:

$$(1) \lim_{x \rightarrow 2} \frac{|x-2|}{x-2} = ? \quad \text{We claim that it DNE.}$$

consider  $x_n = 2 - \frac{1}{n} \xrightarrow[n \rightarrow \infty]{} 2$  &  $y_n = 2 + \frac{1}{n} \xrightarrow[n \rightarrow \infty]{} 2$

$$\text{Then } \frac{|x_n - 2|}{x_n - 2} = \frac{-(2 - \frac{1}{n} - 2)}{2 - \frac{1}{n} - 2} = -1, \text{ whereas}$$

$$\frac{|y_n - 2|}{y_n - 2} = \frac{2 + \frac{1}{n} - 2}{2 + \frac{1}{n} - 2} = 1; \quad 1 \neq -1 \Rightarrow \lim \text{ DNE}$$

by Cor. 4.2.5.

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(2) Infinite limits:

$$\lim_{x \rightarrow c} f(x) = \infty \quad \text{if } \forall M > 0 \exists \delta > 0 \text{ s.t.}$$

$$0 < |x - c| < \delta \Rightarrow f(x) > M$$

Consider  $\lim_{x \rightarrow 0} \frac{1}{x^2}$ . Let  $M > 0$ , and choose  $\delta = \sqrt{\frac{1}{M}}$

$$0 < |x - 0| < \delta = \sqrt{\frac{1}{M}} \Rightarrow x^2 < \frac{1}{M} \Rightarrow$$

$$\frac{1}{x^2} > M \text{ as desired.}$$

(3) Squeeze Theorem (Exer. 4.2.11):

Let  $f, g, h : A \rightarrow \mathbb{R}$  w/  $f(x) \leq g(x) \leq h(x)$   
 $\forall x \in A$ . If  $\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = L$  for a limit pt  $c$  of  $A$ , then  $\lim_{x \rightarrow c} g(x) = L$  as well.

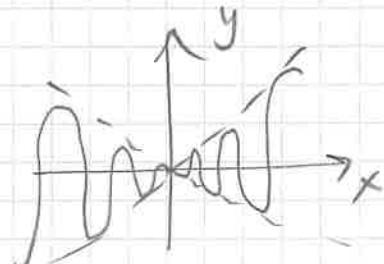
Ex: Show  $\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$

$$-1 \leq \sin \frac{1}{x} \leq 1 \quad \forall x$$

$$\Rightarrow -x \leq x \sin \frac{1}{x} \leq x \quad \forall x > 0$$

$$\text{or } -x \geq x \sin \frac{1}{x} \geq x \quad \forall x < 0$$

In either case,  $\lim_{x \rightarrow 0} x = \lim_{x \rightarrow 0} (-x) = 0 \Rightarrow$   
 $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$  by Squeeze Thm.



One-Sided limits:  $\lim_{x \rightarrow c^+} f(x)$ ,  $\lim_{x \rightarrow c^-} f(x)$

See Exer. 4.2.10  $\rightarrow$  HW