

§ 4.3 Continuous Functions

Def: Continuity

A function $f: A \rightarrow \mathbb{R}$ is continuous at a pt. $c \in A$ if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall x \in A, |x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.

Note: if f is cont. $\forall a \in A \Rightarrow f$ is cont. on A .

Interesting: if c is a limit pt. of A , then

Def. $\Rightarrow \lim_{x \rightarrow c} f(x) = f(c)$. If c is an isolated pt. of A , then $\lim_{x \rightarrow c} f(x)$ DNE, but def can still be applied! \rightarrow Exer. 4.3.5:

if c is an isolated pt. of A , then $\exists \delta > 0$ s.t. $V_\delta(c) \cap A = \{c\}$. Hence $\forall \varepsilon > 0 \exists \delta' > 0$ and \exists the only pt, namely c , w/ $|x - c| < \delta'$. Thus, $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \varepsilon$. So, f is cont. at c . ($f(x) = x$ on N is cont. $\forall x$)

Theorem 4.3.2 (Characterization of Continuity)

Let $f: A \rightarrow \mathbb{R}$, $c \in A$. The func. f is cont. at c iff one of the following conditions is met:

- (def)(1) $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$.
- (2) $\forall V_\varepsilon(f(c))$, $\exists V_\delta(c)$ s.t. $\forall x \in A, x \in V_\delta(c) \Rightarrow f(x) \in V_\varepsilon(f(c))$.

$$(3) \quad \forall (x_n) \underset{n \rightarrow \infty}{\rightarrow} c \quad (x_n \in A), \quad f(x_n) \underset{n \rightarrow \infty}{\rightarrow} f(c) \quad (2)$$

If c is a limit pt. of A , then (1) - (3) are equivalent to:

$$(4) \quad \lim_{x \rightarrow c} f(x) = f(c)$$

"Proof:"

(1) def. of continuity.

(2) rewording of (1)

(3) is equivalent to (1) via Thm (4.2.3)
(sequential criterion for func. limits)

(4) discussed. \square

Note: statement (3) can be used to show a func. is not cont. at a pt.

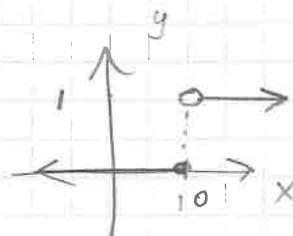
Corollary (4.3.3) (Criterion for Discontinuity)

Let $f: A \rightarrow \mathbb{R}$, and let $c \in A$ be a limit pt. of A . If $\exists (x_n) \subset A, (x_n) \rightarrow c$ s.t. $f(x_n) \not\rightarrow f(c)$, then f is not cont. at c .

Example: Let $f(x) = \begin{cases} 0, & x \leq 10 \\ 1, & x > 10 \end{cases}$

Note $f(10) = 0$. Consider $x_n = 10 + \frac{1}{n} \underset{n \rightarrow \infty}{\rightarrow} 10$
since $\forall n \quad 10 + \frac{1}{n} > 10 \Rightarrow f(x_n) = 1 \rightarrow 1 \neq 0$

Thus, f is discontinuous at 10.



Theorem 4.3.4 (Algebraic Continuity Theorem) (3)

Let $f, g : A \rightarrow \mathbb{R}$

be cont. functions at a pt. $c \in A$. Then the following functions are also cont. at c :

$$(1) kf(x) \quad \forall k \in \mathbb{R}$$

$$(2) f(x) + g(x)$$

$$(3) f(x)g(x)$$

$$(4) \frac{f(x)}{g(x)}, \text{ provided } g(x) \neq 0.$$

Proof from Cor. 4.2.4 & Thm. 4.3.2.

Alg. Limit Thm. Charact. of Cont.
for func. limits

Examples :

(1) polynomials are continuous everywhere.

- a) first, show $g(x) = x$ & $h(x) = k$ are cont.
- b) second, use Thm. 4.3.4

(2) rational func's are cont. wherever they are defined.

$$(3) \text{ Consider } f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Set $c=0$. Note $|f(x) - f(0)| = |x \sin\frac{1}{x} - 0| = |x \sin\frac{1}{x}| \leq |x|$. Then $\forall \varepsilon > 0 \exists \delta = \varepsilon$ s.t. $\forall x, |x - 0| = |x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon$.

(Note $f(x)$ is called a continuous extension of the discontin. function $x \sin\frac{1}{x}$)

(4) Exer. 4.3.8

Show $f(x) = \sqrt{x}$ is cont. on $A = \{x \in \mathbb{R}, x \geq 0\}$.

Proof. Pick $\epsilon > 0$. We want to show that at any $c \in A$, $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$. $(c > 0)$

(a) If $c = 0 \Rightarrow |f(x) - f(c)| = \sqrt{x} < \epsilon$ when $x < \epsilon^2$, i.e., we choose $\delta = \epsilon^2$ to make it happen: $|x - 0| < \delta = \epsilon^2 \Rightarrow |\sqrt{x} - 0| < \epsilon$.

(b) If $c > 0 \Rightarrow$ to estimate $|f(x) - f(c)| = |\sqrt{x} - \sqrt{c}|$, we use the trick:

$$\begin{aligned} |\sqrt{x} - \sqrt{c}| &= |\sqrt{x} - \sqrt{c}| \left(\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \right) \\ &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \leq \frac{|x - c|}{\sqrt{c}} < \epsilon, \text{ i.e. } |x - c| < \underbrace{\epsilon \sqrt{c}}_{\delta}. \end{aligned}$$

Then $\forall \epsilon > 0 \exists \delta = \epsilon \sqrt{c} > 0$ s.t. $\forall x$, $|x - c| < \delta$

$$= \epsilon \sqrt{c} \Rightarrow |\sqrt{x} - \sqrt{c}| < \frac{\epsilon \sqrt{c}}{\sqrt{c}} = \epsilon. \quad \square$$

- To cover more functions, we need:

Theorem (4.3.9) Composition of Continuous Functions.

Let $f: A \rightarrow \mathbb{R}$, $g: B \rightarrow \mathbb{R}$, and the range $f(A) = \{f(x): x \in A\} \subset B$, s.t. the composition $(g \circ f)(x) = g(f(x))$ is defined on A . If f is cont. at $c \in A$, g is cont. at $f(c) \in B$, then $g \circ f$ is cont. at c . (Proof: Exer. 4.3.3)

Examples: $\sqrt{3x^2 + 50}$, $\frac{1}{\sqrt{x}}$, etc.