

§ 4.4. Continuous Functions on Compact Sets,

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Let $f: A \rightarrow R$ be a function.

If $B \subset A \Rightarrow f(B) = \{f(x) : x \in B\}$ is the range (or image) of f over set B .

Q: are the properties of B preserved when B is mapped to $f(B)$ via continuous f ?
(open / closed / compact / bdd, ...)

- B -open $\Rightarrow f(B)$ -open? False:
let $f(x) = \underbrace{x^2}_{\text{cont.}}$, $B = \underbrace{(-1,1)}_{\text{open}} \Rightarrow f(B) = \underbrace{[0,1]}_{\text{not open}}$
 - B -closed $\Rightarrow f(B)$ -closed? False:
(constructing a counterexample is harder)
let $g(x) = \underbrace{\frac{1}{1+x^2}}_{\text{continuous}}$, $B = \underbrace{[0, \infty)}_{\text{closed}}$ ($\forall x \in B$ is a limit pt)
 $= \{x: x \neq 0\}$ also not bdd...
 - then $g(B) = (0,1]$, not closed!
 - B is compact (closed & bdd) $\Rightarrow f(B)$ is compact.

Theorem (4.4.1) (Preservation of Compact sets)

Let $f: A \rightarrow \mathbb{R}$ be continuous on A . If $B \subset A$ is compact, then $f(B)$ is compact as well.

Proof: Let (y_n) be a seq. in $f(B)$. To prove $f(B)$ is compact, we need to find

a subseq $(y_{n_k}) \subset f(B)$, s.t. $y_{n_k} \rightarrow$ limit in $f(B)$. (2)

Since $(y_n) \subset f(B)$, then $\forall n \in \mathbb{N}, \exists x_n \in B$ w/ $f(x_n) = y_n$. Thus, (x_n) is a seq. in B . We know that B is compact $\Rightarrow (x_n)$ has a subseq. $(x_{n_k}) \rightarrow x \in B$. Since f is cont. on $B \Rightarrow (x_{n_k}) \xrightarrow{k \rightarrow \infty} x \Rightarrow \underbrace{f(x_{n_k})}_{y_{n_k}} \xrightarrow{k \rightarrow \infty} f(x)$. From $x \in B \Rightarrow f(x) \in f(B)$. Hence, $f(B)$ is compact. □

Theorem (4.4.2) Extreme Value Theorem

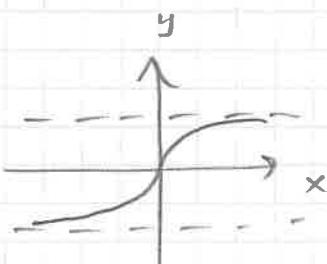
If $f: A \rightarrow \mathbb{R}$ is cont. on a compact set $A \subset \mathbb{R}$, then f attains its max & min value. That is, $\exists x_1, x_2 \in A$ s.t. $f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in A$.

Proof: A - compact, f - cont. $\Rightarrow f(A)$ is compact. \rightarrow Exer. 3.3.1 $\Rightarrow \sup f(A)$ & $\inf f(A)$ are in $f(A)$. Thus, $\exists x_1, x_2 \in A$ w/ $f(x_2) = \sup f(A) = \max f(A)$ & $f(x_1) = \inf f(A) = \min f(A)$. □

Note: If P is a perfect set (closed, no isolated pts), then $f(P)$ is not necessarily perfect even if f is cont. Example: $\arctan x$ maps $\mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$

↙
perfect
(closed,
+pt. is
a lim. pt.)

↙
open,
not closed



Uniform Continuity

We know that polynomials are cont. functions.
 However, let us take a look at the following examples using ϵ - δ def. of continuity:

Example (4.4.3)

(1) $f(x) = 3x+1$ is cont. $\forall c \in \mathbb{R}$, because:

$$\forall \epsilon > 0 \quad \exists \delta = \frac{\epsilon}{3} > 0, \text{ s.t. } \forall x, |x-c| < \delta = \frac{\epsilon}{3},$$

$$|f(x) - f(c)| = |(3x+1) - (3c+1)| = 3|x-c| < 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

Note: $\delta = \frac{\epsilon}{3} \quad \forall c.$

(2) $g(x) = x^2$ is cont. $\forall c \in \mathbb{R}$.

Note: $|g(x) - g(c)| = |x^2 - c^2| = |x-c||x+c|$

$$\begin{aligned} |x+c| &\leq |x| + |c| \leq (|c|+1) + |c| \\ |x|-|c| &\leq |x-c| < \delta \leq 1 \end{aligned}$$

need
a bound.

We can have this if we insist that $\delta \leq 1$

$$\Rightarrow x \in V_\delta(c) \quad (\text{recall example w/ } \begin{matrix} x^2 \rightarrow 4 \\ x \rightarrow 2 \end{matrix})$$

$$\begin{aligned} \text{Now: } \epsilon > 0. \quad \text{If } \delta = \min \left\{ 1, \frac{\epsilon}{2|c|+1} \right\}, \\ \text{then if } |x-c| < \delta \Rightarrow |x^2 - c^2| &= |x-c||x+c| \\ &\leq \frac{\epsilon}{2|c|+1} (2|c|+1) = \epsilon. \end{aligned}$$

Compare to (1): $\delta = \min \left\{ 1, \frac{\epsilon}{2|c|+1} \right\}$
 depends on $c!$

Def. Uniform Continuity. \rightarrow defined on a set! (4)

A func. $f: A \rightarrow \mathbb{R}$ is uniformly continuous on A if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $\forall x, y \in A, |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

it may depend on \subseteq as well.

f cont. $\forall c \in A \leftarrow \forall \varepsilon > 0 \exists \delta > 0 (\delta = \delta(\varepsilon))$ s.t. $\forall x, |x-c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

• Uniform continuity is much stricter:

$\forall \varepsilon > 0 \exists \delta$ that works simultaneously $\forall c \in A$.
(a single δ)

Note: f is not uniformly cont. on $A \nRightarrow f$ is not cont. on A ; it means that $\exists \bar{\varepsilon} > 0$ for which no single $\delta > 0$ is suitable $\forall c \in A$.

Theorem (4.4.5) (Sequential Criterion for Absence of Uniform Continuity)

A func. $f: A \rightarrow \mathbb{R}$ fails to be uniformly cont. on A iff $\exists \bar{\varepsilon} > 0$ & $(x_n), (y_n) \in A$

with: $|x_n - y_n| \rightarrow 0$ but $|f(x_n) - f(y_n)| > \bar{\varepsilon}$.

Proof: text p. 132,
(uses negation of def.) \rightarrow not uniformly cont.

Example (4.4.3 (a)): $g(x) = x^2$ on \mathbb{R} .

Take $x_n = n, y_n = n + \frac{1}{n}$, then it is continuous

$$|n - n - \frac{1}{n}| = |\frac{1}{n}| = \frac{1}{n} \rightarrow 0, \text{ but}$$

$$|n^2 - (n + \frac{1}{n})^2| = |n^2 - n^2 - 2n + \frac{1}{n^2}| = |2 - \frac{1}{n^2}| \rightarrow 2,$$

i.e. $\exists \bar{\varepsilon} = 1$ s.t. $|g(x_n) - g(y_n)| \geq 1 \quad \forall n. \quad n \rightarrow \infty$

Interesting: if we restrict domain of g

to $[-1, 1]$, then $|x+y| \leq 2 \quad \forall x, y \Rightarrow$

$$\forall \varepsilon > 0 \exists \delta = \frac{\varepsilon}{2} \text{ s.t. } \forall x, y \in [-1, 1], |x-y| < \delta = \varepsilon/2 \Rightarrow$$

See also
ex. 4.4.6

$$|x^2 - y^2| = |x-y| |x+y| \leq \frac{\epsilon}{2} \cdot 2 = \epsilon. \quad \text{Note: } [-1,1] \text{ is a compact set.} \quad (5)$$

Theorem (4.4.7) A func. continues on a compact set K is uniformly cont. on K .

Proof: Let $f: K \rightarrow \mathbb{R}$ be cont. $\forall x \in K$.

Assume, by contradiction, that f is not uniformly cont. on K . Then, by Thm. 4.4.5,

$\exists \bar{\epsilon} > 0$ & $(x_n), (y_n) \subset K$ s.t. $|x_n - y_n| \rightarrow 0$, but $|f(x_n) - f(y_n)| \geq \bar{\epsilon} \forall n \in \mathbb{N}$. Note K is compact $\Rightarrow \exists (x_{n_k}) \xrightarrow[n \rightarrow \infty]{K} x \in K$. Let (y_{n_k}) be a subseq. of (y_n) that consists of terms corresponding to the terms in (x_{n_k}) . By the Alg. Limit Thm,

$$\lim_{k \rightarrow \infty} (y_{n_k}) = \lim_{k \rightarrow \infty} \underbrace{(y_{n_k} - x_{n_k}) + x_{n_k}}_{\substack{\downarrow \\ f\text{-cont.}}} = 0 + x = x.$$

That is, $(x_{n_k}) \rightarrow x$, $(y_{n_k}) \rightarrow x \xrightarrow[0]{f\text{-cont.}} f(x_{n_k}) \rightarrow f(x)$, $f(y_{n_k}) \rightarrow f(x) \Rightarrow \lim_{k \rightarrow \infty} (f(x_{n_k}) - f(y_{n_k})) = 0$ which

contradicts the assumption that $|f(x_n) - f(y_n)| \geq \bar{\epsilon} \forall n \in \mathbb{N}$. Thus, f must be uniformly cont. on K

□