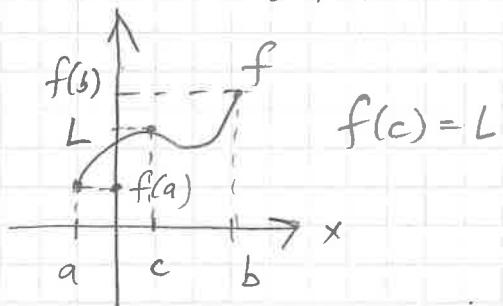


§ 4.5 The Intermediate Value Theorem. ①

Theorem (4.5.1) \leftarrow (IVT)

Let $f: [a, b] \rightarrow \mathbb{R}$ be continuous. If $L \in \mathbb{R}$ satisfies $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then $\exists c \in (a, b)$ where $f(c) = L$.



Note:

Proof was not offered until 1817 \rightarrow Bolzano.

• Application of IVT: proving the existence of roots.

Example: Let $f(x) = x^3 - x + 1$ on $[-2, 2]$. Since $f(2) = 7 > 0$, $f(-2) = -5 < 0$, then by the IVT, $\exists c \in (-2, 2)$ s.t. $f(c) = 0$.

(Numerical analysis: the bisection method.)

Another example: Recall proving that $\sqrt{2}$ exists? Using IVT: if $f(x) = x^2 - 2$ on $[-1, 2]$ then $f(-1) = -1 < 0$, $f(2) = 2 > 0 \Rightarrow \exists c \in (-1, 2)$ s.t. $f(c) = c^2 - 2 = 0$, i.e. $c^2 = 2$.

Proof of the IVT:

Proof F using AoC \leftarrow if $A \neq \emptyset$ & A bdd above
 $\Rightarrow \exists \sup A$ (or l.u.b.)

Consider first the case where f is cont. w/
 $f(a) < 0 < f(b)$ & show $f(c) = 0$ for $c \in (a, b)$.

Let $K = \{x \in [a, b] : f(x) \leq 0\}$. Note $a \in K$,

so $K \neq \emptyset$ & K is bdd above $\Rightarrow \sup K$ exists.

Note that $c = \sup K$ is what we need: (2)

$f(c) = 0$. Why? If $f(c) > 0$, say $f(c) = \varepsilon_0^{>0}$, then since f is cont. $\Rightarrow \exists \delta_0 > 0$ s.t. $x \in V_{\delta_0}(c)$ $\Rightarrow f(x) \in V_{\varepsilon_0}(f(c)) \Rightarrow f(x) > 0$ and $x \notin K$ for all $x \in V_{\delta_0}(c)$. This means that among u.b.'s

$$\begin{array}{c} V_{\delta_0}(c) \\ \underbrace{(-\delta_0, c + \delta_0)} \\ c - \delta_0 \quad c \quad c + \delta_0 \\ \text{x} \notin K \text{ here!} \end{array}$$

of K we have c , $c + \delta_0$, and $c - \delta_0$. Since $c - \delta_0 < c \Rightarrow c \neq \sup K$, so we have a contradiction.

Thus, $f(c)$ cannot be > 0 .

Similarly, one can show $f(c)$ cannot be < 0 .

Thus, $f(c) = 0$. This proves the IVT for the case $f(a) < 0 < f(b)$. For a general case, use $h(x) = f(x) - L$ ($f(a) < L < f(b)$) which is cont. and $h(a) < 0 < h(b)$.

Applying the IVT for the special case, $\exists c \in (a, b)$ s.t. $h(c) = f(c) - L = 0 \Rightarrow f(c) = L$.

□

Proof II: uses NIP

The Intermediate Value Property:

f has the intermediate value property on $[a, b]$ if $\forall x < y$ in $[a, b]$ and L between $f(x)$ & $f(y)$, $\exists c \in (x, y)$ w/ $f(c) = L$.

Note: every cont. func. on $[a, b]$ has IV property.

Careful: if f has IV prop. $\not\Rightarrow$ cont. Example:

$$g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

not cont. at 0, but has IV prop. on $[0, 1]$

More on Continuity

(some material
from §4.6)

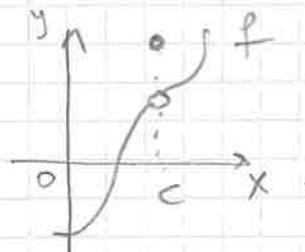
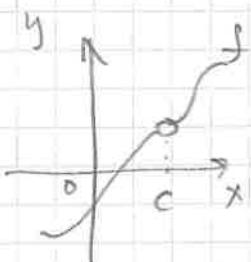
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Recall if c is a limit pt. of A and $f: A \rightarrow \mathbb{R}$,
then f is cont. at c if $\lim_{x \rightarrow c} f(x) = f(c)$.

Types of discontinuities:

1) $\lim_{x \rightarrow c} f(x)$ exists, but $\lim_{x \rightarrow c} f(x) \neq f(c) \Rightarrow$

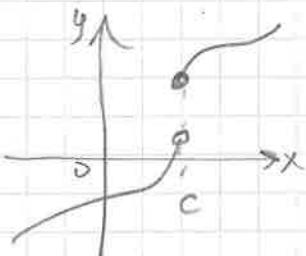
f has a removable discontinuity at c ("hole")



Ex:

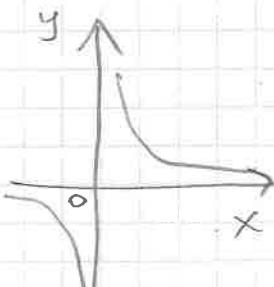
$f(x) = \frac{x^2 - 4}{x - 2}$ has
"hole" at $x=2$

2) $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x) \Rightarrow f$ has a jump
discontinuity at c



Ex: $f(x) = \frac{|x-2|}{x-2}$ has a jump
at $x=2$

3) $\lim_{x \rightarrow c} f(x)$ DNE for any other reason \Rightarrow
 f has an essential discontinuity at c



Ex: $g(x) = \frac{1}{x}$ has essential
discont. at 0