

## Chapter 5

## The Derivative

①

- Questions:
- what is the derivative and differentiability of a function?
  - are cont. functions differentiable?
  - are diff. func's continuous?
  - what can we say about the derivative as a function?

(Read discussion in §5.1)

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### §5.2 Derivative & the IV Property.

(We will consider functions on intervals.)

↳ connected sets

Def: Differentiability

Let  $g: A \rightarrow \mathbb{R}$  be a function defined on an interval  $A$ . Given  $c \in A$ , the derivative of  $g$  at  $c$  is the limit

$$g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}, \text{ provided it exists.}$$

We say  $g$  is differentiable at  $c$ . If  $g'$  exists  $\forall c \in A$ , we say  $g$  is differentiable on  $A$ .

(Note:  $g'(c) = \lim_{\Delta x \rightarrow 0} \frac{g(c + \Delta x) - g(c)}{\Delta x}$ )

Examples:

(1)  $f(x) = x^2$ . Let  $c \in \mathbb{R}$ .

$$f'(c) = \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \rightarrow c} \frac{(x - c)(x + c)}{x - c} = 2c$$

More generally, if  $f(x) = x^n$  ( $n \in \mathbb{N}$ ), then using the identity  $x^n - c^n = (x - c)(x^{n-1} + cx^{n-2} + \dots + c^{n-1})$ ,

(2)

we can find  $f'(c) = \lim_{x \rightarrow c} \frac{x^n - c^n}{x - c}$

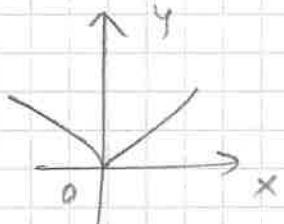
$$= \lim_{x \rightarrow c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \dots + c^{n-1})$$

$$= \underbrace{c^{n-1} + c^{n-1} + \dots + c^{n-1}}_{1 \text{ times}} = nc^{n-1}. \quad (\text{Power rule})$$

(2) If  $f(x) = |x|$ , then  $f$  is not diff. at 0.

Indeed:

if  $x > 0$ , then



$$g'(0) = \lim_{x \rightarrow 0^+} \frac{|x-0|}{x-0} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1$$

If  $x < 0$ , then

$$g'(0) = \lim_{x \rightarrow 0^-} \frac{|x-0|}{x-0} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$$

Thus,  $g'(0)$  DNE.

- Last example shows: continuity  $\not\Rightarrow$  differentiability.

On the other hand:

Theorem (5.2.3) If  $g: A \rightarrow \mathbb{R}$  is diff. at  $c \in A$ , then  $g$  is continuous at  $c$  as well.

Proof: Given  $g'(c) = \lim_{x \rightarrow c} \frac{g(x)-g(c)}{x-c}$ , we

$$\text{have } \lim_{x \rightarrow c} (g(x)-g(c)) = \lim_{x \rightarrow c} \left( \frac{g(x)-g(c)}{x-c} \right) (x-c)$$

$$= g'(c) \lim_{x \rightarrow c} (x-c) = g'(c) \cdot 0 = 0 \Rightarrow \lim_{x \rightarrow c} g(x) = g(c)$$

(Recall  $A$  is an interval, so  $\forall c \in A$  is a limit pt) □

### Theorem (S.2.4) Algebraic Differentiability

Let  $f$  &  $g$  be functions  $A \rightarrow \mathbb{R}$ ,  
differentiable at  $c \in A$ . Then,

Theorem.

$$(1) (f+g)'(c) = f'(c) + g'(c)$$

$$(2) (kf)'(c) = kf'(c) \quad \forall k \in \mathbb{R}$$

$$(3) (fg)'(c) = f'(c)g(c) + f(c)g'(c)$$

$$(4) (f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}, \quad g(c) \neq 0$$

Proof: (1), (2) DIY

$$(3) \text{ Note that } \frac{(fg)(x) - (fg)(c)}{x-c} =$$

$$= \frac{f(x)g(x) + f(x)g(c) - f(x)g(c) - f(c)g(c)}{x-c}$$

$$= f(x) \left[ \underbrace{\frac{g(x) - g(c)}{x-c}}_{\text{diff & cont. at } c} \right] + g(c) \left[ \underbrace{\frac{f(x) - f(c)}{x-c}}_{\text{diff & cont. at } c} \right]$$

so  $\lim_{x \rightarrow c} f(x) = f(c)$

$$\lim_{x \rightarrow c} \frac{(fg)(x) - (fg)(c)}{x-c} = f(c)g'(c) + g(c)f'(c).$$

(4) use similar approach as in (3) or prove  
based on the next result  $\leftarrow$  left as exercise.

Chain Rule: composition of 2 diff. func's  
is also differentiable  $\square$

## Theorem (5.2.5)

(Chain Rule)

(4)

Let  $f: A \rightarrow \mathbb{R}$  and  $g: B \rightarrow \mathbb{R}$  satisfy  $f(A) \subset B$   
 so  $g \circ f$  is defined:  $g \circ f: A \rightarrow \mathbb{R}$ . If  $f$  is  
 diff. at  $c \in A$  &  $g$  is diff. at  $f(c) \in B$ , then  
 $g \circ f$  is diff. at  $c$  and  $(g \circ f)'(c) = g'(f(c)) f'(c)$ .

Proof: Since  $g$  is diff. at  $f(c)$ :

$$\exists g'(f(c)) = \lim_{y \rightarrow f(c)} \frac{g(y) - g(f(c))}{y - f(c)}. \text{ Note: } g(y) - g(f(c))$$

$$(*) = \left[ \frac{g(y) - g(f(c))}{y - f(c)} \right] (y - f(c)). \text{ This holds } \forall y \in B,$$

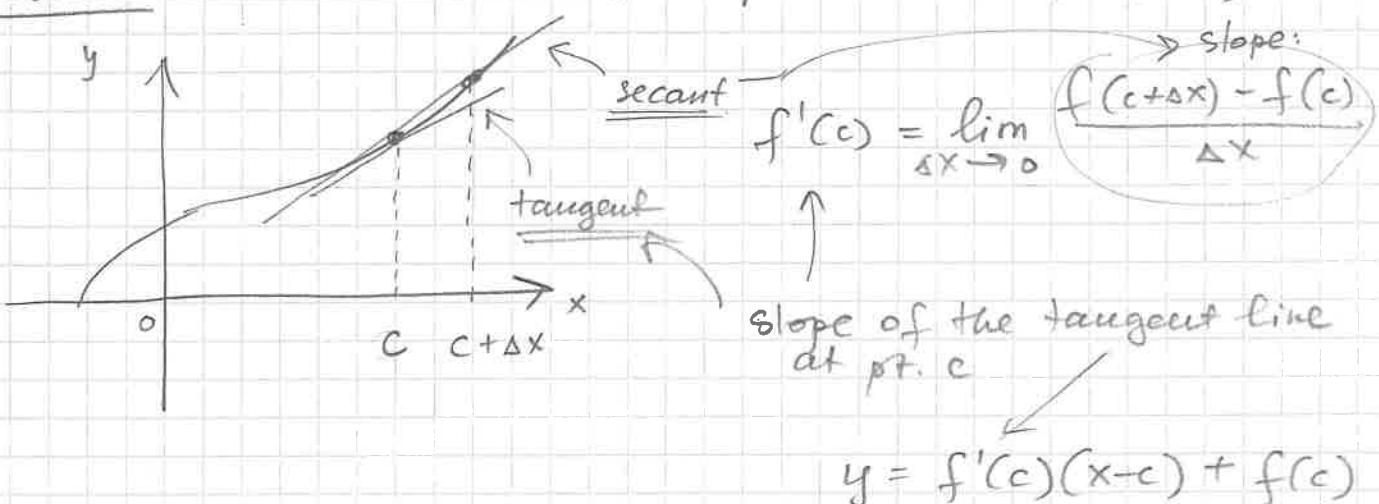
including  $y = f(c)$ :  $\left\{ \begin{array}{l} \text{if we declare} \\ \text{that } d(f(c)) = g'(f(c)) \\ \text{then } d \text{ is cont. at } f(c) \end{array} \right\} \quad \begin{array}{l} y = f(x) \\ \text{for } x \in A. \end{array}$

$$\text{if } x \neq c, \text{ then } (*) \Rightarrow (\text{dividing by } x - c) \\ (g \circ f)(x) = \frac{g(f(x)) - g(f(c))}{x - c} = \left[ \frac{g(f(x)) - g(f(c))}{f(x) - f(c)} \right] \left[ \frac{f(x) - f(c)}{x - c} \right]$$

Taking limit as  $x \rightarrow c$  and applying Alg. Limit  
 Thus:

$$\boxed{(g \circ f)'(c) = g'(f(c)) f'(c)} \quad \square$$

Recall: Geometric interpretation of  $f'(x)$



Differentiability:  $f$  has  $f'(c)$ .

(5)

Also:  $\Delta f(c) = f(\underbrace{c + \Delta x}_x) - f(c) = f'(c) \Delta x + \epsilon \Delta x$

or  $f(x) - f(c) = f'(c)(\underbrace{x - c}_x) + \epsilon(x - c)$

$\epsilon \rightarrow 0$  as  
 $\Delta x \rightarrow 0$

$f(x) \approx \underbrace{f'(c)(x - c)}_{\text{near } c} + f(c)$

tangent line eqn...

Also  $dy = f'(c) dx$

"infinitesimal"  $\rightarrow$  differential of  $f$  at  $c$ , infinitely small change in  $f$   $\rightarrow$  differential  $x$  (infinitely small change in  $x$ )

$\rightarrow (f/g)' = (f'g - g'f)/g^2$

• To prove (4) quotient rule (Thm. 5.2.4):

1) prove that  $f'(c) = -\frac{1}{c^2}$  for  $f(x) = \frac{1}{x}$ .

(2) Combine 1) w/ Chain rule

1)  $\rightarrow f'(c) = \lim_{x \rightarrow c} \frac{\frac{1}{x} - \frac{1}{c}}{x - c} = \lim_{x \rightarrow c} \frac{(c-x)}{xc(x-c)} = \lim_{x \rightarrow c} -\frac{1}{xc}$   
 $= -\frac{1}{c^2}$ .

2) let  $h(x) = \frac{1}{x}$ . Then for a diff. func.  $g(x)$ ,

$(h \circ g)(x) = h(g(x)) = \frac{1}{g(x)}$  and

$\left(\frac{1}{g(x)}\right)' = -\frac{1}{(g(x))^2} \cdot g'(x)$ . Then, using (3)  
Chain rule

from Thm. (5.2.4) (product rule), we have:

$$\begin{aligned} \left(\frac{f}{g}\right)'(x) &= [f(x)(h \circ g)(x)]' = f'(x)(h \circ g)(x) + f(x)(h \circ g)'(x) \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad (g(x) \neq 0) \end{aligned}$$

□

(6)

### Theorem (5.2.6) (Interior Extremum Theorem)

Let  $f$  be diff. on  $(a, b)$ . If  $f$  attains max at  $c \in (a, b)$  ( $f(c) \geq f(x) \forall x \in (a, b)$ ) then  $f'(c) = 0$ . (Likewise, if  $f$  attains min at  $c \in (a, b)$  then  $f'(c) = 0$ )

→ Used as a tool in solving optimization problems.

### Theorem (5.2.7) (Darboux's Theorem)

→ 1842-1917  
French mathematician

If  $f$  is diff. on  $[a, b]$ , and if  $y$  is s.t.  $f'(a) < y < f'(b)$  or  $f'(b) < y < f'(a)$ , then  $\exists c \in (a, b)$  where  $f'(c) = y$ .

"If  $f'$  attains two distinct values, then it must attain every value in between".