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§ 6.2. Sequence of Functions: Uniform Convergence.

Def: (Pointwise Convergence)

Let $f_n : A \rightarrow \mathbb{R}$ be a function $\forall n \in \mathbb{N}$.

The sequence (f_n) converges pointwise on A to a function f if, $\forall x \in A$, the sequence of real numbers $(f_n(x))$ converges to $f(x)$.

We write: $f_n \rightarrow f$, $\lim_{n \rightarrow \infty} f_n = f$

Examples:

$$(1) \quad f_n(x) = \frac{x^2 + nx}{n} \quad \forall x \in \mathbb{R}$$

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \frac{x^2 + nx}{n} = \lim_{n \rightarrow \infty} \frac{x^2}{n} + x = x, \text{ i.e.}$$

$f_n \rightarrow x \quad \forall x \in \mathbb{R}$. (Converges pointwise)

$$(2) \quad g_n(x) = x^n \text{ on } [0, 1]$$

$$0 \leq x < 1, \quad x^n \xrightarrow[n \rightarrow \infty]{} 0$$

$$x = 1, \quad x^n \xrightarrow[n \rightarrow \infty]{} 1$$

$$\text{Thus, } g_n \xrightarrow[n \rightarrow \infty]{} g = \begin{cases} 0, & x \in [0, 1) \\ 1, & x = 1 \end{cases}$$

not continuous

$$(3) \quad h_n(x) = x^{1 + \frac{1}{2n-1}} \text{ on } [-1, 1]$$

$$x \in [-1, 1] : \lim_{n \rightarrow \infty} h_n(x) = \lim_{n \rightarrow \infty} \left(x \cdot x^{\frac{1}{2n-1}} \right)$$

$$= x \lim_{n \rightarrow \infty} x^{\frac{1}{2n-1}} = |x| \leftarrow \text{(pointwise)}$$

$$(\text{why?} \rightarrow) \text{sign}(x) = \begin{cases} x > 0, & +1 \\ x < 0, & -1 \\ x = 0, & 0 \end{cases}$$

continuous $\forall x$,
not diff. at 0

(Note $\frac{1}{2n-1}$ is odd root $\sqrt[2n-1]{x}$)

(2)

Let (f_n) be a seq. of cont. functions on $A \subset \mathbb{R}$, and $f_n \xrightarrow{n \rightarrow \infty} f$ (conv. pointwise). Is f continuous? Nope → recall last examples.
necessarily

If we use definition, we need $\forall \epsilon > 0 \exists \delta > 0$
s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$

Note:

$$|f(x) - f(c)| = |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)|$$

$$\leq \underbrace{|f(x) - f_n(x)|}_{\substack{\uparrow \\ \text{problem:}}} + \underbrace{|f_n(x) - f_n(c)|}_{\substack{\text{can make} \\ < \frac{\epsilon}{3} \\ (\text{for this } n)}} + \underbrace{|f_n(c) - f(c)|}_{\substack{\text{can make} \\ < \frac{\epsilon}{3} \text{ for some } N}}$$

(from continuity of f_n) ($f_n \rightarrow f \ \forall x$)
for $\delta > 0$, $|x - c| < \delta$

all $x : |x - c| < \delta$

↑
values of x depend on δ which depends on N
We cannot just change N each time for different values of x (x is not fixed the way c is!)

See example w/ $g_n(x) = x^n$ on $[0, 1]$ again.

$$x = \frac{1}{2} \quad |g_n\left(\frac{1}{2}\right) - g\left(\frac{1}{2}\right)| \xrightarrow{\quad} \begin{cases} 0, & 0 < x < 1 \\ 1, & x = 1 \end{cases}$$

$$= \left| \left(\frac{1}{2}\right)^n - 0 \right| < \frac{1}{3}, \text{ for } n \geq 2$$

$$x = \frac{9}{10} \quad |g_n\left(\frac{9}{10}\right) - g\left(\frac{9}{10}\right)|$$

$$= \left| \left(\frac{9}{10}\right)^n - 0 \right| < \frac{1}{3} \text{ for } n \geq 11$$

(3)

We need uniform convergence for f_n !

Def.: Let (f_n) be a seq. of func's defined on $A \subset \mathbb{R}$. Then, (f_n) converges uniformly on A to a limit f (defined on A), if $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N \wedge x \in A$.

→ Compare to the def. of pointwise convergence:

Def. (again): Let (f_n) be a seq. of func's defined on $A \subset \mathbb{R}$. Then, (f_n) converges pointwise on A to a limit f (defined on A) if $\forall \varepsilon > 0$ and $x \in A$, $\exists N \in \mathbb{N}$ (depending on x) s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$.

"Uniformly": $\exists N$ is chosen for all x in A simultaneously!

Examples:

(1) $g_n(x) = \frac{1}{n(1+x^2)}$ Note: $\forall x \in \mathbb{R}, \lim_{n \rightarrow \infty} g_n(x) = 0$,
so $g_n \rightarrow 0$ pointwise on \mathbb{R} . Is the convergence

uniform? Consider $|g_n(x) - g(x)| = \left| \frac{1}{n(1+x^2)} - 0 \right| \leq \left| \frac{1}{n} \right| = \frac{1}{n}$, that is, $\forall \varepsilon > 0, \exists N > \frac{1}{\varepsilon}$

s.t. $\forall n \geq N \Rightarrow |g_n(x) - g(x)|$ does not depend on x !

$\leq \frac{1}{n} < \varepsilon$. Thus, $g_n \rightarrow 0$ converges uniformly!

$$(2) f_n(x) = \frac{x^2 + nx}{n} \xrightarrow[n \rightarrow \infty]{} x \quad (\text{shown earlier}) \quad (4)$$

However, on \mathbb{R} , the convergence is not uniform:

$$|f_n(x) - f(x)| = \left| \frac{x^2 + nx}{n} - x \right| = \frac{x^2}{n}$$

If we want $\frac{x^2}{n} < \varepsilon$, then we have to choose $N > \frac{x^2}{\varepsilon}$ s.t. $|f_n(x) - f(x)| < \varepsilon$.

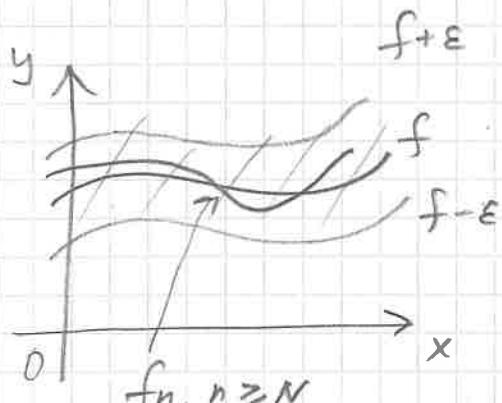
Observe that if $x \in [-a, a]$, then $\frac{x^2}{n} \leq \frac{a^2}{n}$ and $N > \frac{a^2}{\varepsilon}$ independently of x , i.e.

$f_n(x)$ converges to x uniformly on $[-a, a]$

Geometrically:

$f_n \rightarrow f$ uniformly:

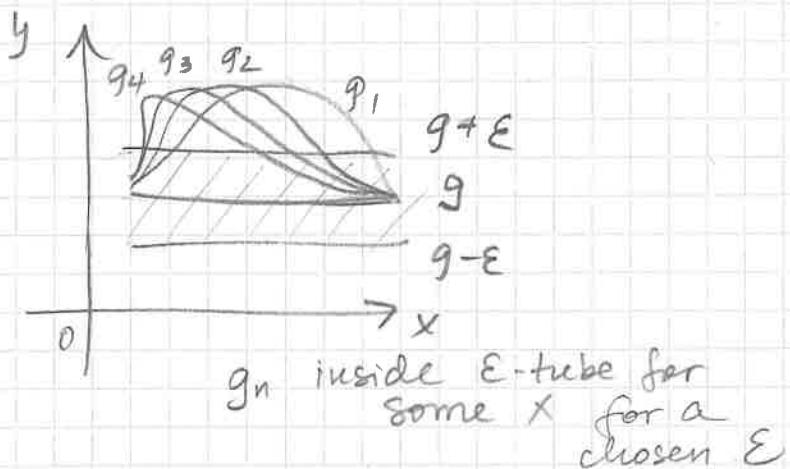
$\exists N$, after which each f_n sits inside ε -tube, $\forall x$!



$g_n \rightarrow g$

not uniformly:

(only pointwise)



(5)

Cauchy Criterion

Recall : (x_n) converges $\Leftrightarrow (x_n)$ is Cauchy,
 i.e. $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $\forall m, n \geq N \Rightarrow |x_n - x_m| < \epsilon$.
 (the limit is not mentioned explicitly)

Theorem (6.2.5) (Cauchy Criterion for Uniform

A sequence of func's (f_n) on a set $A \subset \mathbb{R}$ Convergence
 converges uniformly on A if and only if $\forall \epsilon > 0 \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ & $x \in A$.

Proof : (\Rightarrow) (f_n) converges uniformly on A :

Let $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\forall \epsilon > 0 \exists N$ s.t.

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall n \geq N \text{ & } x \in A.$$

Given $m, n \geq N$,

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \epsilon \quad \forall x \in A. \\ &\qquad\qquad\qquad < \epsilon/2 \qquad\qquad\qquad < \epsilon/2 \end{aligned}$$

(\Leftarrow) Now we assume that $\forall \epsilon > 0 \exists N$ s.t.
 $|f_n(x) - f_m(x)| < \epsilon \quad \forall m, n \geq N \text{ & } x \in A$. Prove
 $f_n(x)$ converges uniformly.

Candidate for the limit? Notice that
 $\forall x \in A$, from the assumption, seq. $(f_n(x))$ is
 Cauchy. Cauchy sequences converge \Rightarrow
 $\exists f(x) = \lim_{n \rightarrow \infty} f_n(x)$. Thus, $f_n \xrightarrow{n \rightarrow \infty} f$ pointwise,

since we can apply the Cauchy Criterion ⑥ to $\forall x \in A$.

Now let $\epsilon > 0$. By assumption, $\exists N$ s.t. $-\epsilon < f_n(x) - f_m(x) < \epsilon \quad \forall m, n \geq N, x \in A$.

By the Alg. Limit Thm,

$$\lim_{m \rightarrow \infty} (f_n(x) - f_m(x)) = f_n(x) - f(x)$$

for
 & fixed n
 $\forall x \in A$

and by the Order Limit Thm,

$$-\epsilon < f_n(x) - f(x) < \epsilon \quad \forall n \geq N, \forall x \in A.$$

That is, $\forall \epsilon > 0 \exists N$ s.t. $|f_n(x) - f(x)| < \epsilon$
 $\forall n \geq N, \forall x \in A$.

□

Continuity Revisited:

Theorem (6.2.6) (Continuous Limit Theorem)

Let (f_n) be a sequence of functions defined on $A \subset \mathbb{R}$ that converges uniformly to f on A .

If $\forall n$, f_n is cont. at $c \in A \Rightarrow f$ is cont. at c .

Proof: Fix $c \in A$. Let $\epsilon > 0$. Choose N s.t.

$$|f_N(x) - f(x)| < \epsilon/3 \quad \forall x \in A. \text{ Also, } \exists \delta > 0$$

uniform convergence

s.t. $|f_N(x) - f_N(c)| < \epsilon/3 \quad \forall x, |x - c| < \delta$ (\Leftarrow cont. of f_N)

Thus, $|f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. So, f is cont at c ! of f_N convergence □