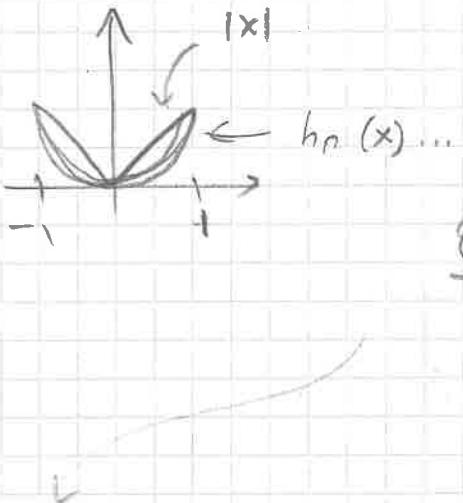


### ①

## § 6.3. Uniform Convergence & Differentiation.

Recall:  $h_n(x) = x^{1 + \frac{1}{2n-1}}$   
on  $[-1, 1]$

$\forall x \in [-1, 1]$ ,  $h_n(x) \xrightarrow[n \rightarrow \infty]{} |x|$  ← cont., but  
not differentiable  
(pointwise)



Note that  $h_n(x)$  are diff.!

Q: Would that be enough to say that if  $h_n \rightarrow h$  uniformly and  $h_n(x)$  are diff.  $\forall n$ , then  $h(x)$  is diff. too?

+ theorem (6.3.1) (Differentiable Limit Theorem)

Let  $f_n \rightarrow f$  pointwise on  $[a, b]$  & let  $f_n$  be differentiable  $\forall n$ . If  $f'_n \rightarrow g$  uniformly on  $[a, b]$ , then  $f$  is diff. and  $f' = g$ .

Theorem (6.3.2) Let  $(f_n)$  be a sequence of diff. func's defined on the interval  $[a, b]$ , and let  $(f'_n)$  converge uniformly on  $[a, b]$ . If  $\exists x_0 \in [a, b]$  where  $f_n(x_0)$  is convergent, then  $(f_n)$  converges uniformly on  $[a, b]$ .

Theorem (6.3.3) Let  $(f_n)$  be a sequence of diff. func's defined on  $[a, b]$ , and let

do not need  
converg. of  $f_n$   
at  $x_0$

$(f_n')$  converge uniformly to  $g$  on  $[a, b]$ . ②  
 If  $\exists x_0 \in [a, b]$  for which  $f_n(x_0)$  is convergent, then  $(f_n)$  converges uniformly and the limit function  $f = \lim_{n \rightarrow \infty} f_n$  is diff. &  $f' = g$ .

(Thm. 6.3.1 + 6.3.2)

Note: in Thm 6.3.2 we only use  $f_n \rightarrow f$  for some  $x_0$  (in 6.3.1 we have  $f_n \rightarrow f \quad \forall x \in [a, b]$ )

That is, uniform conv. of  $(f_n')$  is strong enough to show that  $(f_n)$  conv. uniformly  
 nearly

Example:  $g_n(x) = \frac{x^n}{n} \xrightarrow[n \rightarrow \infty]{} 0$  on  $[0, 1]$

Let  $\epsilon > 0$ . Consider  $|\frac{x^n}{n} - 0| = |\frac{x^n}{n}| \leq \frac{1}{n}$

Pick  $N > 1/\epsilon$  (does not depend on  $x$ )  $\forall x \in [0, 1]$

Then  $\forall \epsilon > 0 \exists N > 1/\epsilon$  s.t.  $\forall n \geq N > 1/\epsilon \Rightarrow$

$|\frac{x^n}{n} - 0| \leq \frac{1}{n} < \epsilon \quad \forall x \in [0, 1]$ . So,  $g_n(x) \rightarrow 0$

uniformly on  $[0, 1]$ .

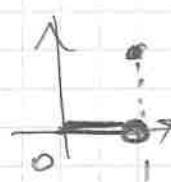
Since  $g(x) = 0 \quad \forall x \in [0, 1] \Rightarrow g$  is diff. and  $g' = 0$ .

$g_n'(x) = \frac{nx^{n-1}}{n} = x^{n-1} \xrightarrow[n \rightarrow \infty]{} \begin{cases} 0, & x \neq 1 \\ 1, & x = 1 \end{cases}$  on  $[0, 1]$

This convergence is not uniform:

because if it were, then by Thm 6.2.6  
 (Cont. Limit Thm), we'd have continuous  
 $\lim_{n \rightarrow \infty} g_n'(x)$ , but  $\lim_{n \rightarrow \infty} g_n'(x) = h(x) = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$  is  
 discontinuous!

(3)



Also,  $g' = 0 + h(x)$   
 on  $[0, 1]$

Example:  $h_n(x) = \frac{\sin(nx)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{} 0 \quad \forall x$  by  
 squeeze Thm.

Consider

$$|h_n(x) - 0| = \left| \frac{\sin(nx)}{\sqrt{n}} \right| \leq \frac{1}{\sqrt{n}} < \epsilon$$

Given  $\epsilon > 0$ ,  $\exists N > 1/\epsilon^2$  (does not depend on  $x$ ) want

and  $\forall n \geq N \Rightarrow |h_n(x) - 0| < \epsilon \quad \forall x \in \mathbb{R}$ .

Thus,  $h_n(x) \xrightarrow[n \rightarrow \infty]{} 0$  uniformly.

However,  $h_n'(x) = \sqrt{n} \cos(nx)$  diverges  $\forall x \neq \frac{\pi}{2} + k\pi$

Note: uniform conv. of a sequence of func's  
 does not imply anything particular about  
 the behavior of seq. of their derivatives!