

## § 6.4 Series of Functions

①

Def: Let  $f_n \forall n \in \mathbb{N}$  &  $f$  be functions  $A \rightarrow \mathbb{R}$ .

The infinite series  $\sum_{n=1}^{\infty} f_n(x) = f_1(x) + f_2(x) + f_3(x) + \dots$  converges pointwise on  $A$  to  $f(x)$  if the sequence

$S_k(x)$  of partial sums,  $S_k(x) = \sum_{l=1}^k f_l(x)$   
 $= f_1(x) + f_2(x) + \dots + f_k(x)$  converges pointwise

to  $f(x)$ .  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly on  $A$  to  $f$ , if  $S_k(x)$  converges uniformly on  $A$  to  $f$ .

$$S_1(x) = f_1(x)$$

$$S_2(x) = f_1(x) + f_2(x)$$

⋮

$$S_k(x) = f_1(x) + f_2(x) + \dots + f_k(x)$$

⋮

We write:

$$f = \sum_{n=1}^{\infty} f_n \text{ or}$$

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

(being explicit about the type of convergence!)

Examples:

$$\textcircled{1} \quad \sum_{n=0}^{\infty} \frac{x^{2n}}{(1+x^{2n})} = \frac{1}{2} + \frac{x^2}{1+x^2} + \frac{x^4}{1+x^4} + \dots$$

$$\textcircled{2} \quad \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \dots$$

$$\textcircled{3} \quad \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} = \sin x + \frac{\sin(2x)}{8} + \frac{\sin(3x)}{27} + \dots$$

Theorem (6.4.2) (Term-by-Term Continuity)

Let  $f_n$  be cont. func's on a set  $A$  ( $\forall n$ ). Assume  $\sum_{n=1}^{\infty} f_n$  converges uniformly on  $A$  to a function  $f$ .

Then  $f$  is continuous on  $A$ . (2)

Proof: Apply Thm. (6.26)  $\left( \begin{array}{l} (f_n) \xrightarrow{\text{unif.}} f, f_n \text{ cont.} \\ \Rightarrow f \text{ cont.} \end{array} \right)$   
to the partial sums  $S_k(x) = f_1(x) + \dots + f_k(x)$   
Since  $S_k(x) \rightarrow f$  & each  $S_k$  is cont.  
(as a sum of uniformly cont. func's), thus,  $f$  is cont. □

Theorem (6.4.3) (Term-By-Term Differentiability)

Let  $f_n$  be diff. func's on an interval  $A$  &  
assume  $\sum_{n=1}^{\infty} f_n'(x)$  converges uniformly to  
a limit  $g(x)$  on  $A$ . If  $\exists x_0 \in A$  where  
 $\sum_{n=1}^{\infty} f_n(x_0)$  converges (p-w), then the series  
 $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly to a diff. func.  
 $f(x)$  w/  $f'(x) = g(x)$  on  $A$ . That is,

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \& \quad f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

Proof: Apply the stronger Theorem 6.3.3  
to the partial sums  $S_k = f_1 + f_2 + \dots + f_k$ .  
(Recall that  $S_k' = f_1' + f_2' + \dots + f_k'$  by Thm. 5.2.4  
(Alg. Diff. Thm.) □

Theorem (6.4.4) (Cauchy Criterion for  
Uniform Convergence of Series).

A series  $\sum_{n=1}^{\infty} f_n$  conv. uniformly on  $A \subset \mathbb{R}$  if  
and only if  $\forall \epsilon > 0 \exists N \in \mathbb{N}$  s.t.

$|f_{m+1}(x) + f_{m+2}(x) + \dots + f_n(x)| < \epsilon$  (3)  
 whenever  $n > m \geq N$  and  $x \in A$ .

Proof: here we again consider  $S_1(x), S_2(x), \dots$   
 (seq. of part. sums) and use Thm. 6.4.5  $\rightarrow$   
 Cauchy Crit. for Uniform Convergence:  
 $(S_k(x))$  conv. unif.  $\Leftrightarrow \forall \epsilon > 0 \exists N$  s.t.

$$|S_n(x) - S_m(x)| = |(f_1(x) + \dots + f_m(x) + \dots + f_n(x)) - (f_1(x) + \dots + f_m(x))| = |f_{m+1}(x) + \dots + f_n(x)| < \epsilon$$

$\forall n > m \geq N, x \in A. \quad \square$

Corollary (6.4.5): Weierstrass M-Test:

$\forall n \in \mathbb{N}$ , let  $f_n$  be a function  $A \rightarrow \mathbb{R}$ ,  $\leftarrow$  very useful  
 & let  $M_n > 0$  ( $M_n \in \mathbb{R}$ ) is s.t.  $|f_n(x)| \leq M_n$   
 $\forall x \in A$ . If  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n(x)$   
 conv. uniformly on  $A$ .

Proof: Key: use the Cauchy Criterion for series of real #'s (Thm. 2.7.2:  $\sum a_k$  conv iff  $\forall \epsilon > 0 \exists N$  s.t.  $\forall n > m \geq N, |a_{m+1} + \dots + a_n| < \epsilon$ )

Let  $\epsilon > 0$ .  $\sum M_n$  conv  $\Rightarrow \exists N$  s.t.

$\forall n > m \geq N, |M_{m+1} + \dots + M_n| < \epsilon$ . Since

$$|f_{m+1}(x) + \dots + f_n(x)| \leq |f_{m+1}(x)| + \dots + |f_n(x)| \leq$$

$$M_{m+1} + \dots + M_n = |M_{m+1} + \dots + M_n| < \epsilon \Rightarrow \text{by}$$

Thm. 6.4.4,  $\sum f_n$  conv. uniformly.  $\square$

Example: Consider  $\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n}$ . Continuous? (4)

Since  $\left| \frac{\cos(2^n x)}{2^n} \right| \leq \frac{1}{2^n} \forall n$  and  $\sum \frac{1}{2^n}$  conv., then by Weierstrass M-Test,  $\sum \frac{\cos(2^n x)}{2^n}$  converges uniformly. Because  $\forall n$   $\frac{\cos(2^n x)}{2^n}$  is cont., then by Thm 6.4a (Term-by-Term Continuity),  $\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{2^n} = g(x)$  is cont. func.

Example: Let  $f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^3}$ .

Show  $f(x)$  is diff. &  $f'(x)$  is cont.

Note:  $\frac{d}{dx} \left( \frac{\sin(kx)}{k^3} \right) = \frac{\cos(kx)}{k^2}$ . Since

$\left| \frac{\cos(kx)}{k^2} \right| \leq \frac{1}{k^2}$  &  $\sum \frac{1}{k^2}$  conv., then by

the W. M-Test,  $\sum \frac{\cos(kx)}{k^2}$  conv. uniformly.   
 derivative

Since for  $x=0$ ,  $\sum \frac{\sin(0)}{k^3} = 0 \Rightarrow$  the series  $\sum \frac{\sin(kx)}{k^3}$  converges at  $x=0$ . Thus, by

Thm. 6.4.3,  $\sum \frac{\sin(kx)}{k^3} = f(x)$  is a diff. func. &

$f'(x) = \sum \frac{\cos(kx)}{k^2}$  (which is also cont. by Thm. 6.4a)