Chapter 4 Solution of a Single Nonlinear Equation in One Unknown.

Example for motivation:

Consider population growth model:

$$\frac{dP(t)}{dt} = \underset{\substack{\downarrow \\ \text{constant birth rate}}}{\lambda} P(t) \qquad \left(\begin{matrix}\text{rate of change} \\ \text{of } P\end{matrix} \sim P\right)$$

$\Rightarrow P(t) = P_0 e^{\lambda t}$, where $P_0 = P(t_0)$ - initial population.

This is an "ideal" model.

If we add an immigration factor:

$$\frac{dP}{dt} = \lambda P + \nu \quad \Rightarrow \quad P(t) = P_0 e^{\lambda t} + \frac{\nu}{\lambda}\left(e^{\lambda t} - 1\right)$$

Suppose: $P_0 = 1{,}000{,}000$, $\nu = 435{,}000$, $P(1) = 1{,}564{,}000$

Question: what is λ? Equation (nonlinear)

$$1{,}564{,}000 = 1{,}000{,}000\, e^{\lambda} + \frac{435{,}000}{\lambda}\left(e^{\lambda} - 1\right)$$

can be solved <u>numerically</u> $\Rightarrow$ root-finding problem.

§ 4.1 Bisection method.

Solve nonlinear equation $f(x) = 0$

We assume that f is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$. Then, by the IVT, there must be $x^* \in [a, b]$ s.t. $f(x^*) = 0$.

Bisection method: f - cont, on $[a,b]$,

$sign(f(a)) \neq sign(f(b))$

Let $a_1 = a$, $b_1 = b$. Compute $c_1 = \frac{a_1+b_1}{2}$ and evaluate $f(c_1)$. If $f(c_1) = 0 \Rightarrow$ done!

If not, then either $sign(f(c_1)) = sign(f(a_1))$ or $sign(f(b_1))$. Choose a_2 and b_2 s.t.

$a_2 = a_1$ & $b_2 = c_1$ if $sign(f(a_1)) \neq sign(f(c_1))$

or $a_2 = c_2$ & $b_2 = b_1$, else.

Reapply the process to the interval $[a_2, b_2]$.

$c_2 = \frac{1}{2}(a_2+b_2), \ldots$

Stop when:

$f(c_n) = 0$ or $|f(c_n)| < \varepsilon$ (small)

or $|b_n - a_n| < \delta$ (small) or after a particular number of iterations.

Rate of convergence: (i.e., how fast does the method converge to a solution)

$$|b_k - a_k| = \frac{|b-a|}{2^k} \underset{k\to\infty}{\longrightarrow} 0 \Rightarrow$$

convergence is linear: if $e_k = c_k - x^*$ (root)

then $|e_{k+1}| = |c_{k+1} - x^*| \leq |b_{k+1} - c_{k+1}| \overset{\text{or}}{=}$

$(x^* \in [a_{k+1}, b_{k+1}])$ $\quad c_{k+1} = \frac{1}{2}(a_{k+1}+b_{k+1})$

$$= |c_{k+1} - a_{k+1}| = \frac{1}{2}|b_{k+1} - a_{k+1}| = \frac{1}{2} \cdot \frac{|b_k - a_k|}{2}$$

and $|e_k| = |c_k - x^*| \leq \frac{1}{2}|b_k - a_k|$

Thus, we have: $|e_{k+1}| \approx \frac{1}{2}|e_k|$

$\Rightarrow$ linear convergence, since the error is reduced by a constant factor at each step.

want $|e_k| < \delta$

Convenient: if we choose some $\delta > 0$, then to obtain an interval of size 2δ, we need: $|b_k - a_k| = \dfrac{|b-a|}{2^k} \leq 2\delta \Rightarrow k \geq \log_2\left(\dfrac{|b-a|}{\delta}\right) - 1$

(Note: $|e_k| \leq \frac{|b_k - a_k|}{2}$)

$c_k - x^*$ $\shortparallel$

For example: to get $e_k \approx 10^{-4}$, how many iterations do we need to take?

Consider $f(x)$ Cont. on $[1,2]$, $f(1)\cdot f(2) < 0$.

$$\frac{|2-1|}{2^k} \leq 2\cdot 10^{-4} \iff k \geq \log_2\left(\frac{1}{10^{-4}}\right) - 1$$

$k \geq 12.29 \Rightarrow$ after 13 iterations, the error $\leq 10^{-4}$.

Or: How many iterations would require to reduce the size of $[1,2]$ to 10^{-10}?

$$\frac{|2-1|}{2^k} \leq 10^{-10} \iff 10^{10} \leq 2^k$$

$$\Rightarrow k \geq \log_2(10^{10}) \approx 33.2 \Rightarrow$$

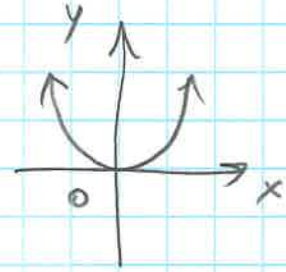
we need 34 iterations.

Note:

(1) In practice, need to find $[a, b]$, s.t. $f(a) \cdot f(b) < 0$

(2) Can't use for non-negative or non-positive functions:

$$f(x) = x^2 \geq 0 \implies$$



$$f(a) \cdot f(b) \geq 0$$

$$\forall a, b.$$