Review of Calculus

① Limits & Continuity → basis for NA techniques analysis.

$\boxed{\text{Def 1}}$ A function f defined on a set X of real numbers has the limit L at $x_0 \in X$, $\lim_{x \to x_0} f(x) = L$, if $\forall \varepsilon > 0$ (for any $\varepsilon > 0$) $\exists \delta > 0$ (there exists $\delta > 0$)

s.t. (such that) $\forall x \in X$ w/ $0 < |x - x_0| < \delta \Rightarrow$

$|f(x) - L| < \varepsilon$ ("$\in$" = "belongs to", or "in")

$\lim_{x \to x_0} f(x)$: "expected value of f at x_0"

$\boxed{\text{Def 2}}$ A function f defined on a set X of real numbers is continuous at $x_0 \in X$ if $\lim_{x \to x_0} f(x) = f(x_0)$

- f is continuous on X if it is continuous $\forall x \in X$.
- $C(X)$ denotes a set of all functions, continuous on X.
- $\mathbb{R} = (-\infty, \infty)$

$\boxed{\text{Def 3}}$ (Limit of a Sequence)

Let $\{x_n\}_{n=1}^{\infty}$ be an infinite sequence of real numbers. Then $\{x_n\}_{n=1}^{\infty}$ converges to x_0 if $\forall \varepsilon > 0$ $\exists N$ s.t. $|x_n - x_0| < \varepsilon$ whenever $n > N$. We write: $\lim_{n \to \infty} x_n = x_0$

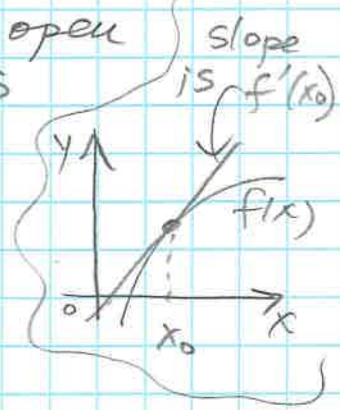
Example: $\{r^n\}_{n=0}^{\infty}$, $r \in \mathbb{R}$

$$\lim_{n\to\infty} r^n = \begin{cases} 0, & |r|<1 \\ 1, & r=1 \end{cases}$$

$\boxed{\text{Def 4}}$ (Differentiability)

Let f be a function defined on an open interval containing x_0. Then f is differentiable at x_0 if

$$\underbrace{f'(x_0)}_{\text{derivative}} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \quad \text{exists.}$$



- f is differentiable on X if $f'(x)$ exists for any $x \in X$.
- If f is differentiable at $x_0 \Rightarrow$ f is continuous at x_0.

Calculus Theorems Used in NA

- Mean Value Theorem (MVT)

If $f \in C[a,b]$ and differentiable on (a,b)
(f is cont. on $[a,b]$)

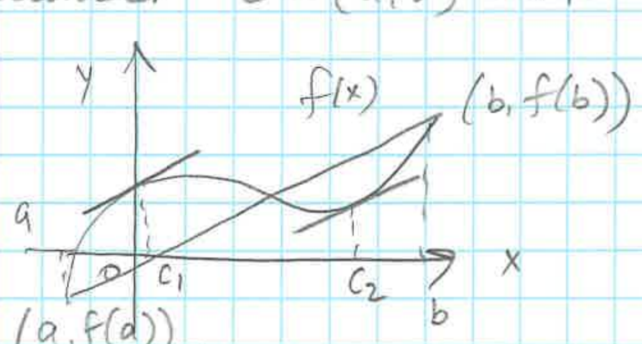
then there exists a number $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

If $f(a)=f(b)$

$\Downarrow$

Rolle's Thm.



Tangents at c_1, c_2 are parallel to the secant line $(a, f(a))$ to $(b, f(b))$

- **Extreme Values:**

Critical pts: x's where $f'(x)=0$

If $f \in C[a,b] \Rightarrow f$ must attain its abs. max & min values; need to check values of f at the endpoints and critical points.

Example: $f(x) = 2 - e^x + 2x$ on $[0,1]$ $(f \in C[0,1])$

$$f'(x) = -e^x + 2 = 0 \Leftrightarrow e^x = 2 \Leftrightarrow x = \ln 2 \approx 0.69$$

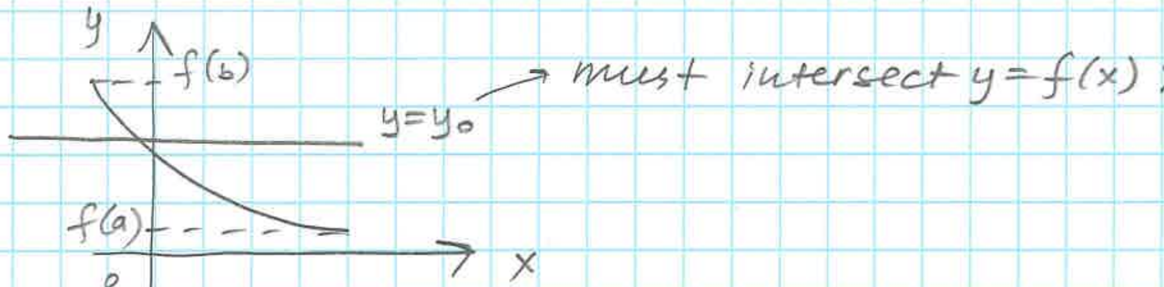
So, $f(0) = 1 \leftarrow$ abs. min

$f(1) = 4 - e \approx 1.28$

$f(\ln 2) = 2\ln 2 \approx 1.39 \leftarrow$ abs. max.

- **Intermediate Value Theorem (IVT)**

If $f \in C[a,b]$ and y_0 lies between $f(a)$ and $f(b)$, then there exists $x \in (a,b)$ s.t. $f(x) = y_0$



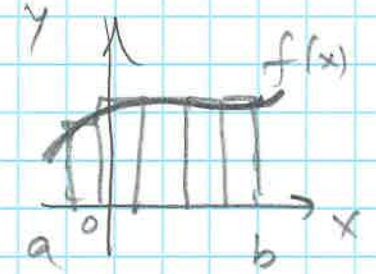
Example: We can use the IVT to show that a function has root(s) in $[a,b]$

$$p(x) = x^5 - 2x^3 + 3x^2 - 1 = 0, \quad [0,1] \quad (p\text{-continuous})$$

Since $p(0) = -1 < 0$ and $p(1) = 1 > 0$, then for $y_0 = 0$ $(-1 < 0 < 1)$, by the IVT, there exists $x^* \in (0,1)$, s.t. $p(x^*) = 0$.

The IVT is used in <u>bisection method.</u> (Chapter 4)

$\boxed{\text{Def 5}}$



$$\int_a^b f(x)\,dx = \lim_{\max(\Delta x_k)\to 0} \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

$a = x_0 < x_1 < x_2 < \dots < x_n = b$
$\Delta x_k = x_k - x_{k-1}, \quad c_k \in [x_{k-1}, x_k]$

→ the (Riemann) definite integral.

- Continuous functions are <u>integrable.</u>

- <u>Taylor Polynomials & Series</u>
 (used extensively in NA)

<u>Taylor's Thm:</u>

Suppose $f \in C^n[a,b]$ (i.e, f has continuous derivatives up to the order n on $[a,b]$) and $f^{(n+1)}$ exists on $[a,b]$. Let $x_0 \in [a,b]$. Then, for every $x \in [a,b]$, there exists a number $\xi(x)$ between x_0 & x with $f(x) = P_n(x) + R_n(x)$

x_0: center

$P_n(x)$: nth Taylor polynomial

$R_n(x)$: remainder or truncation error

Where

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!} \quad \text{and}$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

(ξ is not to be determined explicitly) in NA: looking for a realistic bound for $f^{(n+1)}(\xi)$ (provided $|x-x_0|$ is bounded)

Taking $n \to \infty$, the limit of $P_n(x)$ is an infinite series called the Taylor series for f about x_0. If $x_0 = 0$ then it is called the Maclauren series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

Example: $f(x) = \cos x$, $x_0 = 0$. Determine:

(a) $P_2(x)$ for f about x_0

(b) $P_3(x)$ for f about x_0

Solution: $f(x) \in C^{\infty}(\mathbb{R})$

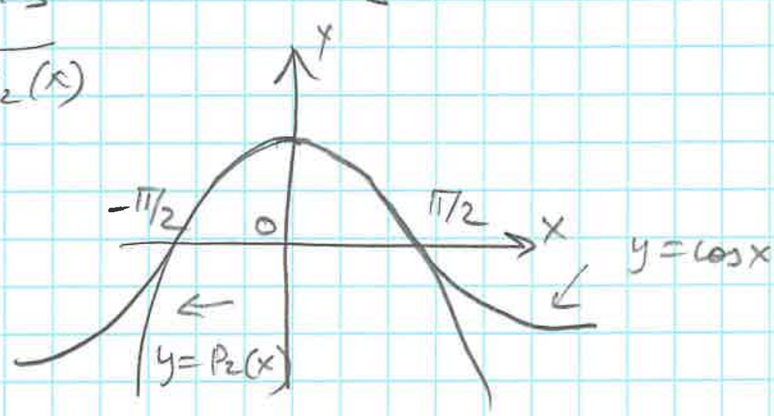
$f' = -\sin x$, $f'' = -\cos x$, $f''' = \sin x$, $f^{(4)}(x) = \cos x = f(x)$ and $f(0) = 1$, $f'(0) = 0$, $f''(0) = -1$, $f'''(0) = 0$.

(a) $n = 2$

$$\cos x = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2}_{P_2(x)} + \underbrace{\frac{f'''(\xi)}{3!}x^3}_{R_2(x)}$$

$$= \underbrace{1 - \frac{1}{2}x^2}_{P_2(x)} + \underbrace{\frac{x^3}{6}\sin\xi}_{R_2(x)} \qquad \xi \text{ between } 0 \,\&\, x.$$

So, $\cos x \approx 1 - \frac{x^2}{2}$ near 0.



Note: if we pick $x = 0.01$, then

$$\cos(0.01) = \underbrace{1 - \frac{1}{2}(0.01)^2}_{= 0.99995} + \underbrace{\frac{1}{6}(0.01)^3 \sin\xi}_{\text{error}}$$

$0 < \xi < 0.01$

$\Rightarrow \cos(0.01) \approx 0.99995$ w/ error

$$|\cos(0.01) - 0.99995| \le 0.1\overline{6} \times 10^{-6}$$

$$(\,|\sin\xi| \le 1\,)$$

Note: the actual error is smaller:

since $|\sin\xi| \le |\xi| \le 0.01$ then the new (better) bound is $0.1\overline{6} \times 10^{-8}$.

(b) $n = 3$.

$$f'''(0) = 0 \Rightarrow \cos x = 1 - \frac{x^2}{2} + \frac{x^3}{3!}\overset{0}{\overset{\shortparallel}{\sin 0}} + \frac{1}{24}x^4 \cos\eta$$

$$\Rightarrow P_2(x) = P_3(x) = 1 - \frac{x^2}{2}$$

η between 0 & x

but the error for $\cos(0.01)$ now is

$$\left|\frac{1}{24}x^4 \cos\eta\right| \le \frac{1}{24}(0.01)^4(1) \approx 4.2 \times 10^{-10}$$

or $|\cos(0.01) - 0.99995| \le 4.2 \times 10^{-10} \Rightarrow$

$P_2(x)$ & $P_3(x)$ provide the same approximation, but $P_3(x)$ gives a better bound!

We can also use the Taylor polynomials to give approximations to integrals:

$$\int_0^{0.1} \cos x \, dx = \int_0^{0.1} \underbrace{\left(1 - \frac{1}{2}x^2\right)}_{P_2(x) = P_3(x)} dx + \int_0^{0.1} \underbrace{\frac{x^4}{24}\cos\eta}_{\substack{\text{error} \\ R_3(x)}} dx$$

$$= \left[x - \frac{1}{6}x^3\right]_0^{0.1} + \frac{1}{24}\int_0^{0.1} x^4 \cos\eta \, dx$$

$$= 0.1 - \frac{0.01^3}{6} + \text{error} \Rightarrow \int_0^{0.1} \cos x \, dx \approx 0.1 - \frac{0.01^3}{6} = 0.0998\overline{3}$$

$$\text{Error} = \frac{1}{24}\left| \int_0^{0.1} x^4 \cos\eta \, dx \right| \le \frac{1}{24}\int_0^{0.1} x^4 \underbrace{|\cos\eta|}_{\le 1} dx$$

$$\le \frac{1}{24}\int_0^{0.1} x^4 \, dx = \frac{1}{24}\frac{(0.1)^5}{5} = 8.\overline{3} \times 10^{-8}$$

(The true value $\int_0^{0.1} \cos x \, dx = \sin x \Big|_0^{0.1} = \sin 0.1$

$\approx 0.0998341664\overline{7}$ & actual error is

8.3314×10^{-8}.)