§8.5. Interpolation at Chebyshev Points.

We learned how to use polynomials to interpolate $y = f(x)$ at given nodes. Some issues with higher-degree polynomials (as in case of $f(x) = \frac{1}{1+x^2}$) can be avoided if we cluster the interpolation nodes near the endpoints of the interval instead of having them equally spaced.

- Chebyshev interpolation pts (also, Gauss-Lobatto pts). $x_j = \cos\left(\frac{\pi j}{n}\right)$, $j = 0, 1, \ldots, n$. ← $x_j \in [-1, 1]$

- Package "chebfun" (MATLAB):

functions are represented by polynomial interpolants at Chebyshev pts. The degree is chosen automatically to attain a level of accuracy close to the machine precision.

Chebfun converts between different representations of the interpolant using the Fast $\underbrace{\text{Fourier Transform}}_{\substack{\text{transforms a function of time}\\ \text{into a function of frequency}}}$ (FFT) → §14.5.1

Note: When x_i, $i = 0, \ldots, n$, are Chebyshev pts, in the error $|f(x) - p(x)| = \frac{1}{(n+1)!} \left| f^{(n+1)}(\xi) \prod_{j=0}^{n} (x - x_j) \right|$

we have $\max_{x \in [-1,1]} \left| \prod_{j=0}^{n} (x - x_j) \right| \le \frac{1}{2^{n-1}}$.

Also, weights in the barycentric formula take form:

$$w_j = \frac{2^{n-1}}{n} \cdot \begin{cases} (-1)^j/2, & j=0 \text{ or } j=n \\ (-1)^j, & \text{otherwise} \end{cases}$$

(Recall: $p(x) = \left(\sum_{i=0}^{n} \frac{w_i}{x-x_i} y_i \right) \Big/ \left(\sum_{i=0}^{n} \frac{w_i}{x-x_i} \right)$, and $\frac{2^{n-1}}{n}$ will be dropped as it appears in both top & bottom of p.)

Theorem (8.5.1) Let f be a continuous func. on $[-1,1]$, and p_n be its degree n polynomial interpolant at the Chebyshev pts $x_j = \cos\frac{\pi j}{n}$, $j = 0, 1, \ldots, n$, and p_n^* be its best approximation among all p of deg n on $[-1,1]$ in the ∞-norm $\|\cdot\|_\infty$. Then the error

$$\| f - p_n \|_\infty \le \left(2 - \frac{2}{\pi} \log n\right) \| f - p_n^* \|_\infty$$

$$\Big(\underbrace{\| f \|_\infty}_{\infty\text{-norm}} = \max \{ |f(x)|, x \in \underset{\substack{\uparrow \\ \text{some set,} \\ \text{for instance,} \\ [-1,1]}}{S} \} \Big)$$

(Thm 8.5.1 means that the interpolant of deg n at Chebyshev pts is close to the best approx. in the ∞-norm, $\| f - p_n \|_\infty = \max_{[-1,1]} | f(x) - p_n(x) |$)