§ 8.6.2. Cubic Spline Interpolation.

<u>Recall:</u>   A piecewise cubic Hermit interpolant has one continuous derivative. To obtain smoother interpolants, consider another option:

• A cubic spline interpolant $S(x)$ is a piecewise cubic that interpolates a func. f at the nodes $x_0, ..., x_n$ (also called knots) and has <u>two</u> continuous derivatives, $S'(x)$ and $S''(x)$.

To get the formula for S, we will solve a global linear system (instead of writing S, S', S'' on each $[x_{i-1}, x_i]$ separately).

• What is the number of equations and unknowns in this system?

 — A cubic S has 4 parameters for each piece on each subinterval $[x_{i-1}, x_i]$ ⟹ <u>$4n$ unknowns</u> (n is # subintervals)

 — From $S(x_i) = f(x_i)$, $i = 0, 1, ..., n$ we get $2n$ equations, since in each subinterval we have to match f at 2 endpoints.

 — Also: S' and S'' must be continuous at the knots. We choose $x_1, ..., x_{n-1}$ knots, leaving x_0, x_n without this requirement, this will allow us to enforce additional conditions on these knots later (results in

different cubic splines). Thus, by requiring s', s'' to be continuous at $x_1, \ldots, x_{n-1}$, we get $2(n-1) = 2n-2$ equations:

$S_1 \quad S_2 \qquad\qquad\qquad S_n$

$x_0 \quad x_1 \quad x_2 \qquad\qquad\qquad x_n$

if s_i is a cubic on the ith subinterval, then:

$$s_i'(x_i) = s_{i+1}'(x_i)$$
$$s_i''(x_i) = s_{i+1}''(x_i)$$
$$i = 1, 2, \ldots, n-1$$

$\underline{\text{Total:}}$ $4n$ unknowns
$(2n-2) + 2n = 4n-2$ equations

$\Downarrow$

2 degrees of freedom.

$\underline{\text{Derivation of } s:}$

Let $z_i = s''(x_i)$, $i = 1, \ldots, n-1$. If we suppose that z_i's are known, then linear s'' must satisfy

$$s_i''(x) = z_{i-1} \frac{x-x_i}{x_{i-1}-x_i} + z_i \frac{x-x_{i-1}}{x_i - x_{i-1}} \quad \text{in } [x_{i-1}, x_i].$$

$\nearrow$ ith piece of our piecewise s''

$\underbrace{\qquad\qquad\qquad}_{\text{Lagrange form}}$

w/ $h = x_i - x_{i-1}$ for all i

So, $s_i''(x) = \frac{z_{i-1}}{h}(x_i - x) + \frac{z_i}{h}(x - x_{i-1})$. Now

integrate twice to get:

$$s_i'(x) = -\frac{z_{i-1}}{h}\frac{(x_i-x)^2}{2} + \frac{z_i}{h}\frac{(x-x_{i-1})^2}{2} + C_i, \text{ and}$$

$$s_i(x) = \frac{z_{i-1}}{h}\frac{(x_i-x)^3}{6} + \frac{z_i}{h}\frac{(x-x_{i-1})^3}{6} + C_i(x-x_{i-1}) + D_i$$

$$\left(s_i'(x) = \int_{x_{i-1}}^{x} s_i''(t)\,dt + C_i \quad \& \quad s_i(x) = \int_{x_{i-1}}^{x} s_i'(t)\,dt + D_i \right)$$

by the FTC

<u>Recall:</u> $S_i(x_{i-1}) = f(x_{i-1}) = f_{i-1}$

for simplicity of notation

$$S_i(x_{i-1}) = z_{i-1} \cdot \frac{h^2}{6} + D_i = f_{i-1} \Rightarrow D_i = f_{i-1} - \frac{z_{i-1} h^2}{6};$$

now from $S_i(x_i) = f(x_i) = f_i$, we have:

$$S_i(x_i) = z_i \frac{h^2}{6} + C_i h + D_i = \frac{z_i h^2}{6} + C_i h + \overbrace{f_{i-1} - \frac{z_{i-1} h^2}{6}}^{D_i} = f_i$$

$$\Rightarrow C_i = \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i) \right].$$

So, $S_i(x) = \frac{1}{h} z_{i-1} \frac{(x_i - x)^3}{6} + \frac{1}{h} z_i \frac{(x - x_{i-1})^3}{6} + $

↗
cubic piece
on the ith interval

$$+ \frac{1}{h}\left[f_i - f_{i-1} + \frac{h^2}{6}(z_{i-1} - z_i) \right](x - x_{i-1})$$

$$+ f_{i-1} - \frac{h^2}{6} z_{i-1}$$

Once we know $z_1, \ldots, z_{n-1} \Rightarrow S$ is known anywhere.
Using continuity of S' (which we did not use yet):

$$S_i'(x_i) = S_{i+1}'(x_i)$$

$$z_i \cdot \frac{h}{2} + \frac{1}{h}(f_i - f_{i-1}) + \frac{h}{6}(z_{i-1} - z_i) = -\frac{h}{2} z_i + \frac{1}{h}(f_{i+1} - f_i)$$
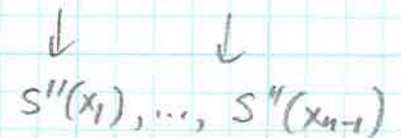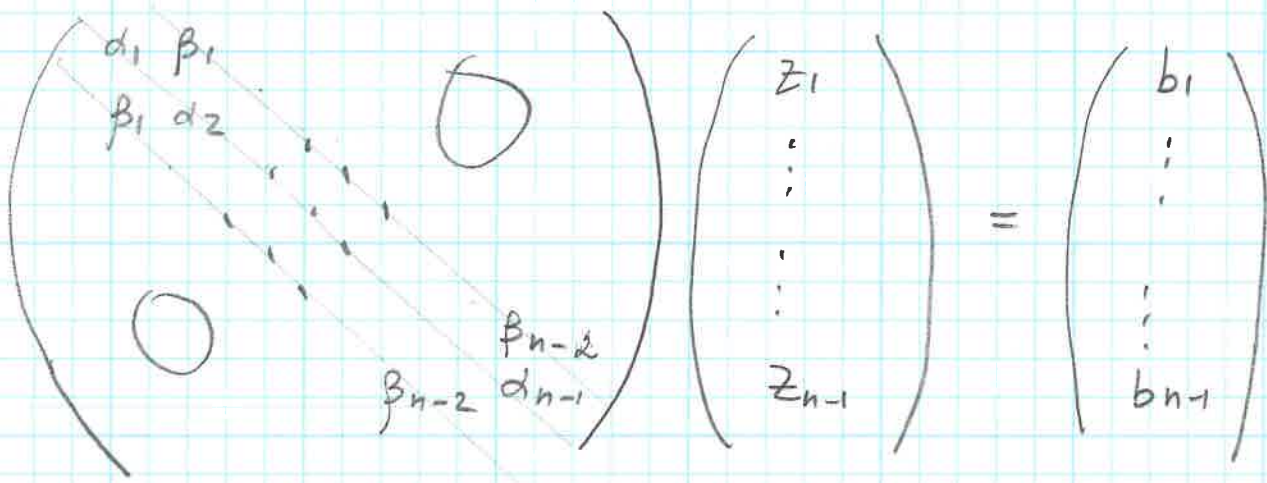$$+ \frac{h}{6}(z_i - z_{i+1})$$

$i = 1, \ldots, n-1.$

Taking z_i's to the L.H.S, and f_i's to the R.H.S:

$$\frac{2h}{3} z_i + \frac{h}{6} z_{i-1} + \frac{h}{6} z_{i+1} = -\frac{2}{h} f_i + \frac{1}{h} f_{i-1} + \frac{1}{h} f_{i+1},$$

$i = 1, \ldots, n-1.$

We can rewrite this as a symmetric
tridiagonal linear system for $z_1, \ldots, z_{n-1}$:

↓   ↓
$S''(x_1), \ldots, S''(x_{n-1})$

$$
\begin{pmatrix}
d_1 & \beta_1 & & & & \\
\beta_1 & d_2 & & & & \\
& & \ddots & & & \\
& & & \ddots & & \\
& & & & \beta_{n-2} & \\
& & & \beta_{n-2} & d_{n-1}
\end{pmatrix}
\begin{pmatrix}
z_1 \\
\vdots \\
\vdots \\
z_{n-1}
\end{pmatrix}
=
\begin{pmatrix}
b_1 \\
\vdots \\
\vdots \\
b_{n-1}
\end{pmatrix}
$$

where $\quad d_i = \dfrac{2h}{3}, \quad \beta_i = \dfrac{h}{6} \quad$ for all i

$$b_i = \frac{1}{h}\left(f_{i+1} - 2f_i + f_{i-1}\right) \quad \text{for} \quad i = 2, \ldots, n-2$$

$$b_1 = \frac{1}{h}\left(f_2 - 2f_1 + f_0\right) - \frac{h}{6} z_0$$

$$b_{n-1} = \frac{1}{h}\left(f_n - 2f_{n-1} + f_{n-2}\right) - \frac{h}{6} z_n$$

<u>Note:</u>   We are free to choose $z_0, z_n \Rightarrow$ different types of splines:

(1) $\boxed{z_0 = z_n = 0}$ $\left(S''(x_0) = S''(x_n) = 0\right)$ gives the <u>natural</u> cubic spline.   Rarely used. Not accurate at x_0, x_n.

(2) z_0 & z_n are chosen s.t. $S'(x_0) = f'(x_0)$
    $S''(x_0) \quad S''(x_n)$ $S'(x_n) = f'(x_n)$
Called the <u>complete</u> cubic spline.
(need to have / approx. $f'(x_0)$ and $f'(x_n)$)

(3) z_0 & z_n are chosen s.t. S''' is continuous
at x_1 and x_{n-1}, i.e. $S_1'''(x_1) = S_2'''(x_1)$ and
                $S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$
(S_i is a piece of S on the ith subinterval)

Called not-a-knot spline (appropriate if
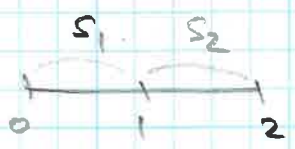$f'(x_0), f'(x_n)$ not
known)

the idea is not to change
S as it crosses x_1 and x_{n-1} => in this case,
" they are not knots " : S''' being a constant
and not-a-knot requirement imposes this!
(i.e., $S_1 = S_2$ and $S_{n-1} = S_n$)

Examples:

(1) Natural spline : $\underset{z_0}{\underline{S''(x_0)}} = \underset{z_n}{\underline{S''(x_n)}} = 0$.

Find it for $f(x) = x^4$ using subintervals
$[0,1], [1,2]$.   => 3 knots: $x_0 = 0, x_1 = 1, x_2 = 2$
                                  $f_0 = 0, f_1 = 1, f_2 = 16$
$n = 2$ intervals

So, we have $z_0 = z_2 = 0$, let's find z_1:

S_1 S_2

$S_1''(0) = 0 = S_2''(2)$

$S_1(1) = S_2(1)$

$S_1'(1) = S_2'(1)$

Using formulas: ($h = 1$)

$d_1 = \dfrac{2h}{3} = 2/3$

$\beta_1 = \dfrac{h}{6} = 1/6$ (don't even need this)

$b_1 = \dfrac{1}{h}(f_2 - 2f_1 - f_0) - \dfrac{h}{6}z_0$
   $= (16 - 2 - 0) - 0 = 14$

So, the system has one equation only; ←

$d_1 z_1 = b_1$ or $\dfrac{2}{3}z_1 = 14 => z_1 = 21$

=> $S_1(x) = 0 + \dfrac{1}{1} \cdot 21 \dfrac{(x-0)^3}{6} + \dfrac{1}{1}(1 - 0 + \dfrac{1^2}{6}(0 - 21))(x - 0)$
   $+ 0 - \dfrac{1^2}{6} \cdot 0 = \dfrac{21 x^3}{6} + (1 - \dfrac{21}{6})x = \dfrac{7}{2}x^3 - \dfrac{5}{2}x$

and $S_2(x) = 1 \cdot 1 \dfrac{(2-x)^3}{6} + 0 + [16 - 1 + \dfrac{1^2}{6}(21 - 0)](x - 1)$
   $+ 1 - \dfrac{1^2}{6} \cdot 21 = \dfrac{(2-x)^3}{6} + (16 - \dfrac{21}{6})(x - 1) = -\dfrac{7}{2}x^3 + 21x^2 - \dfrac{47}{2}x + 7$

Thus, $S(x) = \begin{cases} \frac{7}{2}x^3 - \frac{5}{2}x, & 0 \leq x \leq 1 \\ -\frac{7}{2}x^3 + 21x^2 - \frac{47}{2}x + 7, & 1 \leq x \leq 2 \end{cases}$

(2) Determine the unknown parameters a, b, c, d, so that the function

$$S(x) = \begin{cases} x^2 + x^3 & \overset{S_1(x)}{\leftarrow} & [0,1] \\ a + bx + cx^2 + dx^3, & [1,2] \end{cases} \quad S_2(x)$$

is a cubic spline w/ $S_2'''(x) = 12$.

We should have:

$$S_1(1) = S_2(1), \quad S_1'(1) = S_2'(1), \quad S_2''(1) = S_2''(1)$$

and $S_2'''(x) = 12$

$S_1'(x) = 2x + 3x^2$, $\quad S_2'(x) = b + 2cx + 3dx^2$

$S_1''(x) = 2 + 6x$, $\quad S_2''(x) = 2c + 6dx$, $\quad S_2''' = 6d = 12$

$$\Downarrow$$

$\boxed{d = 2}$

$5 = b + 2c + 3d$

$8 = 2c + 6d$

Since $d = 2 \Rightarrow c = -2 \Rightarrow 5 = b - 4 + 6 \Rightarrow b = 3$

From $S_1(1) = S_2(1) \Rightarrow 2 = a + b + c + d \Rightarrow a = -1$

So, $S_2(x) = -1 + 3x - 2x^2 + 2x^3$ on $[1,2]$.