§ 11.2 One-Step Methods.

(*) IVP $y'(t) = f(t, y(t))$, $\underbrace{t \geq t_0}_{\text{time}}$ } assumption: well-posed
$y(t_0) = y_0$.

<u>Idea</u>: to approximate the solution at time T we start from dividing $[t_0, T]$ into small subintervals with $\underbrace{t_0 < t_1 < \dots < t_{N-1} < t_N}_{\text{nodes or mesh pts}} = T$.

Let y_i = approximation of y at t_i, i.e., $y(t_i)$.
We will replace $\frac{dy}{dt}$ w/ methods from Ch. 9.
A <u>one-step</u> method is one in which y_{k+1} is determined from y_k. (There are also <u>multistep</u> methods, in which we use approximations from time steps $t_{k-1}, t_{k-2}, \dots$, to compute y_{k+1} $\longrightarrow$ not studied here, see <u>Section 11.3.</u>)

<u>One-step methods</u>:

we assume
1) $h = t_{k+1} - t_k$ is uniform $\forall k$
2) $y(t)$ has as many cont. derivatives as needed; this is for error analysis using Taylor's thm.

§ 11.2.1 Euler's Method.

(Leonhard Euler)

Start w/ $y_0 = y(t_0)$,

for $k = 0, 1, \ldots$, set

$$y_{k+1} = y_k + h f(t_k, y_k)$$

(That is, $y(t_{k+1}) \approx y_{k+1}$)

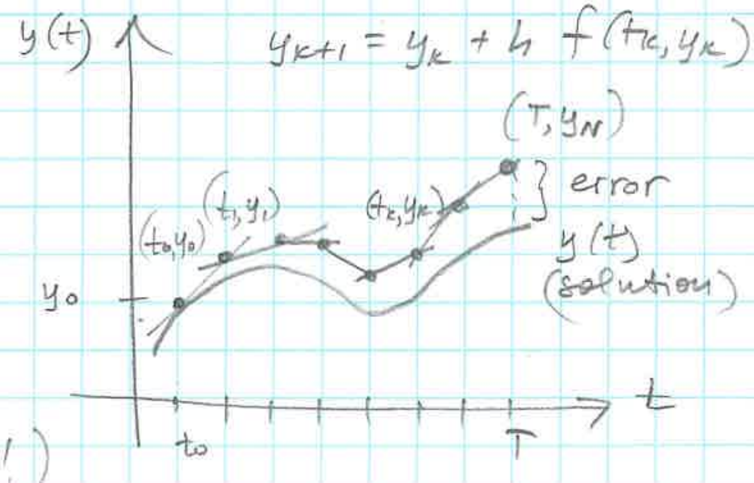
From Taylor's thm:

$$\underbrace{y(t_{k+1})}_{\substack{\text{true}\\ \text{value}\\ \text{at } t_{k+1}}} = \underbrace{y(t_k)}_{\substack{\text{true value}\\ \text{at } t_k}} + \underbrace{(t_{k+1} - t_k)}_{= h} \underbrace{y'(t_k)}_{\substack{= f(t_k, y(t_k))\\ \text{from ODE}}} + \frac{(\overbrace{t_{k+1} - t_k}^{h})^2}{2} y''(\xi_k)$$

$\xi_k \in (t_k, t_{k+1})$

$$= \underbrace{\underset{\downarrow\, y_k}{y(t_k)} + h f(t_k, \underset{\downarrow\, y_k}{y(t_k)})}_{\Rightarrow \text{Euler's}} + \underbrace{\frac{h^2}{2} y''(\xi_k)}_{O(h^2)}$$

Also: $$\underbrace{\frac{y_{k+1} - y_k}{\underset{h}{t_{k+1} - t_k}}}_{\substack{\text{slope}\\ \text{of secant}}} \approx \underbrace{y'(t_k)}_{\substack{\text{slope of } y\\ \text{at } t_k}} \overset{\text{by ODE}}{=} f(t_k, \underset{\downarrow\, y_k}{y(t_k)})$$



Geometric Interpretation:

Tang. lines w/ slopes $f(t_k, y_k)$ are used to approximate sol's.

(Moving from one sol. curve to another, corresponding to different initial conditions!)

Example: IVP $\begin{array}{l} y' = y \\ y(0) = 1 \end{array}$ $\quad (t_0 = 0, y_0 = 1)$

(Solution is $y(t) = e^t$)

The IVP is well-posed: $\underbrace{f(t,y) = y, \ \frac{\partial f}{\partial y} = 1}_{\text{continuous}}$

$\Rightarrow$ f is uniformly Lipschitz continuous since

$$|f(t,y_1) - f(t,y_2)| = |y_1 - y_2| \quad w/ \quad L = 1$$

$\Rightarrow$ (Thm 11.1.4) $\quad |z(t) - y(t)| \le e^{|t|}|\delta|$

for any $z(t)$ s.t. $z(0) = 1 + \delta$

Euler's method for this IVP:

$$\begin{cases} y_0 = 1 \\ y_{k+1} = y_k + h\underbrace{f(t_k, y_k)}_{y_k} = y_k + hy_k = (1+h)\,y_k \end{cases}$$

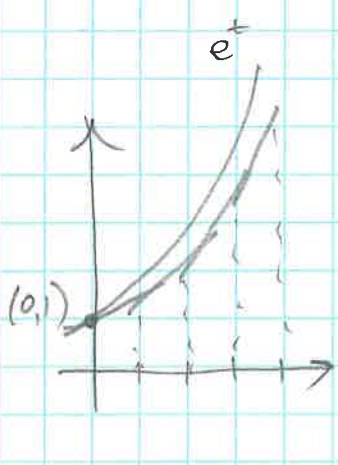
$k = 0, 1, \ldots N$

If $h = \frac{T-0}{N} = \frac{T}{N}$ for some T

$$\Rightarrow \quad y_N = \left(1 + \frac{T}{N}\right)^N \xrightarrow[N \to \infty]{} e^T$$

Let us estimate $y(4) = e^4$ using Euler's method.

If $h = 1$ $\Rightarrow$
$t_0 = 0, \ y_0 = 1$
$t_1 = 1, \ y_1 = (1+1)\,y_0 = 2$
$t_2 = 2, \ y_2 = (1+1)\,y_1 = 4$
$t_3 = 3, \ y_3 = (1+1)\,y_2 = 8$
$t_4 = 4, \ y_4 = (1+1)\,y_3 = 16 \approx y(4) = e^4$



abs. error $= |e^4 - 16| \approx 38.6$

Example w/ system of ODE's:

Lotka - Volterra predator - prey system

Given $R(t_0) = R_0$ (rabbits) & $F(t_0) = F_0$ (foxes)

and ODE's $\begin{cases} R' = (\alpha - \beta F) R \\ F' = (\gamma R - \delta) F \end{cases}$

w/ $\vec{y} = \begin{pmatrix} R(t) \\ F(t) \end{pmatrix}$, $\vec{f}(t, \vec{y}) = \begin{pmatrix} (\alpha - \beta F) R \\ (\gamma R - \delta) F \end{pmatrix}$,

we apply Euler's method:

$$\vec{y}_{k+1} = \vec{y}_k + h \vec{f}(t_k, \vec{y}_k) \quad , \quad k = 0, 1, \ldots$$

Error Discussion: IVP $\circledast$ $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0, \ t \geq t_0 \end{cases}$

① Local Truncation Error (LTE) is the amount by which the true sol. $y(t)$ fails to satisfy the difference equations $\frac{y_{k+1} - y_k}{h} = f(t_k, y_k)$

What does it mean? If we use $y(t_k)$ instead of y_k, then (by Taylor's thm),

$$\frac{y(t_{k+1}) - y(t_k)}{h} = f(t_k, y(t_k)) + \underbrace{\frac{h}{2} y''(\xi_k)}_{= O(h)}$$

$$(t_k \leq \xi_k \leq t_{k+1})$$

$$\left(y(t_{k+1}) = y(t_k) + \underset{\substack{\parallel \\ t_{k+1} - t_k}}{h} y'(t_k) + \frac{h^2}{2} y''(\xi_k) \right)$$

So, LTE is $O(h) \to 0$ as $h \to 0$

(Some sources: LTE is $O(h^2)$ as it is the first neglected term in Taylor's series)

If LTE $\to 0$ as $h \to 0$ $\Rightarrow$ the method is consistent.

2 Global Error (absolute error):

$|y(t_k) - y_k|$ at mesh nodes t_k.

Consider Thm 11.2.1:

Let $y(t)$ be sol. to the IVP (*) and $f(t,y)$ be continuous in the strip $a \le t \le b$, $-\infty < y < \infty$, and uniformly Lipschitz continuous in y.

Let $T \in [a,b]$ (fixed), $t > t_0$, and $h = \frac{T - t_0}{N}$ and

$y_{k+1} = y_k + h\, f(t_k, y_k)$, $k = 0, 1, \dots, N-1$. Then, for any k, s.t. $t_k \in [t_0, T]$, $y_k \to y(t_k)$ as $h \to 0$ and this convergence is uniform:

$\max_k |y(t_k) - y_k| \to 0$ $\left[\forall \varepsilon > 0 \ \exists K_\varepsilon \text{ s.t. } \forall k > K_\varepsilon \text{ and corresp. } t_k \Rightarrow |y(t_k) - y_k| < \varepsilon \right]$

(K_ε depends on ε only)

(Proof: pp. 259-260)

3 Effect of Rounding:

Recall: if h $\underbrace{\text{small}}_{\text{too}}$ $\Rightarrow$ $f'(x) \approx \frac{\overbrace{f(x+h)}^{=0} - f(x)}{h} = 0$

Consider $\underbrace{\tilde{y}_{k+1} = \tilde{y}_k}_{\text{computed sol.}} + h\, f(t_k, \tilde{y}_k) + \underbrace{\delta_k}_{\text{roundoff due to computing } f(t_k, \tilde{y}_k) \text{, multiplying it by } h \text{, and adding } \tilde{y}_k}$

Let $|\delta_k| \le \delta \quad \forall k$

If $\tilde{d}_j = \tilde{y}_j - y_j \leftarrow$ num. sol. using ~~exact~~ arithmetic

computed num. sol.

then $\underbrace{\tilde{y}_{k+1} - y_{k+1}}_{\tilde{d}_{k+1}} = \underbrace{\tilde{y}_k - y_k}_{\tilde{d}_k} + h\left(f(t_k, \tilde{y}_k)\right.$

$\left.- f(t_k, y_k)\right) + \delta_k \implies$

$|\tilde{d}_{k+1}| \le |\tilde{d}_k| + h \underbrace{|f(t_k, \tilde{y}_k) - f(t_k, y_k)|}_{\text{if } f \text{ is Lipschitz continuous} \Rightarrow \le L\underbrace{|\tilde{y}_k - y_k|}_{|\tilde{d}_k|}} + \delta$

$\le |\tilde{d}_k| + Lh|\tilde{d}_k| + \delta = |\tilde{d}_k|(1+Lh) + \delta.$

Using proof of Thm 11.2.1, one can show that $|\tilde{d}_{k+1}| \le e^{(k+1)Lh}|\tilde{d}_0| + \dfrac{e^{(k+1)Lh} - 1}{Lh}\delta \le$

$\le e^{L(T-t_0)}|\tilde{d}_0| + \dfrac{e^{L(T-t_0)} - 1}{Lh}\delta$

$(\tilde{d}_0 = \tilde{y}_0 - y_0)$

Thus, if h is too small: $\frac{1}{h}$ will dominate!

We have to: - choose h small enough to reduce the global error, <u>but</u>

- not too small to avoid big errors due to roundoff!