§4.5 Fixed Point Methods.

Def. A fixed point to a function is a number at which the value of the function does not change, i.e.,

x^* is a fixed pt for $y = \varphi(x)$, if $x^* = \varphi(x^*)$.

Fixed point problem: finding x^* s.t. $x^* = \varphi(x^*)$.

Note: one can translate from a rootfinding problem to a fixed pt. problem and vice versa:

(1) $x = \varphi(x) \iff f(x) = x - \varphi(x) = 0$

(2) $f(x) = 0 \iff x = \varphi(x) = f(x) + x$

Q: Why do we consider fixed pt problems?

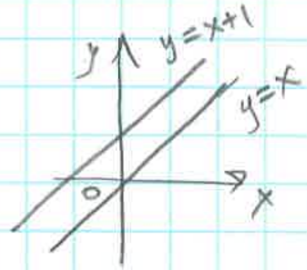
1. Some problems are most naturally expressed as fixed pt problems (data-flow analysis in economics, theory of phase transitions)
2. Sometimes they are easier to solve.
3. What we learn from analysis of $x = \varphi(x)$ helps to find good root-finding strategies.

Examples: Solve $x = \varphi(x)$

(1) $\varphi(x) = x^2 - 3x + 4 \Rightarrow$

$$x = x^2 - 3x + 4$$
$$x^2 - 4x + 4 = 0$$
$$(x-2)^2 = 0$$
$$\boxed{x^* = 2}$$

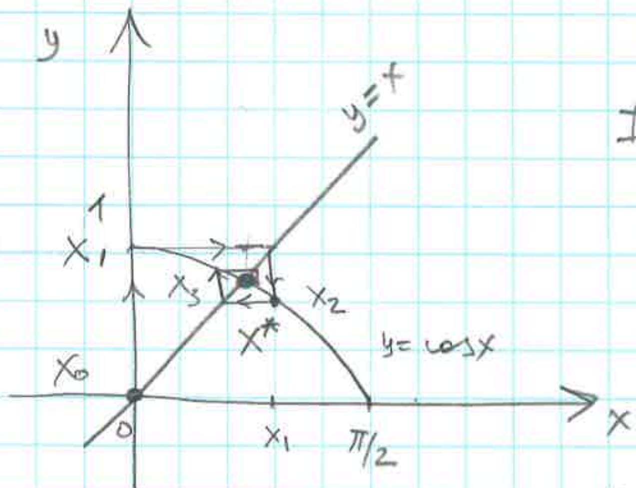
(2) $\varphi(x) = x+1 \Rightarrow x+1 = x$ has no sol's!



parallel lines!

Note: geometrically, x^* s.t. $x^* = \varphi(x^*)$ is the x-coordinate of the point of intersection of the graphs $y = \varphi(x)$ and $y = x$.

(3) $x = \cos x$ on $[0, \pi/2]$



How do we find x^*?

Idea: start at some x_0, then find $x_1 = \cos x_0$
$x_2 = \cos x_1$
. . . .
$x_n = \cos x_{n-1}$

So, we have

$$x^* \approx x_n = \underbrace{\cos(\cos(\cos(\dots}_{n \text{ times!}}(x_0)\dots)))$$

Eventually, $x_n \to x^* = 0.739085133215$ "famous fixed point"

So, it is natural to iterate like this:

$$\underbrace{x_{k+1} = \varphi(x_k), \quad k = 0, 1, 2, \dots}_{\text{fixed-point iteration}} \quad \text{w/ given } x_0 \text{ (initial guess)}$$

Both Newton's method & the constant slope method can be thought as fixed point iterations:

N.M:

$$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_k)}}_{\varphi(x_k)}, \quad k = 0,1,2,\ldots$$

i.e. $\varphi(x) = x - f(x)/f'(x)$

whose fixed pts are being sought

Constant Slope:

$$x_{k+1} = \underbrace{x_k - \frac{f(x_k)}{f'(x_0)}}_{\varphi(x_k)}$$

$k = 0,1,2,$

i.e. $\varphi(x) = x - f(x)/f'(x_0)$

whose fixed points are being sought

Note: $\varphi(x) = x \Leftrightarrow f(x) = 0$

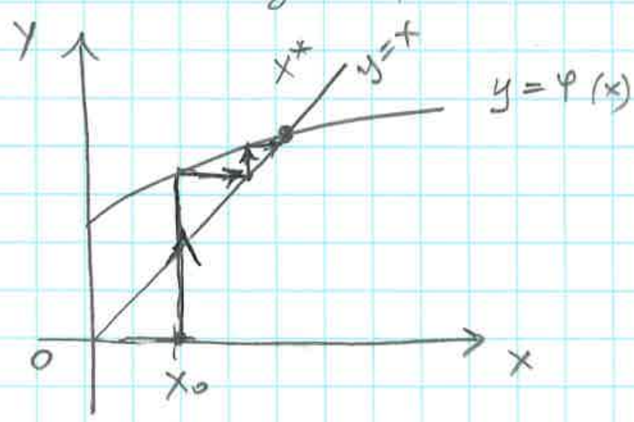
- What about the secant method? – Nope!

Recall: $$x_{k+1} = x_k - \frac{f(x_k)(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})}$$

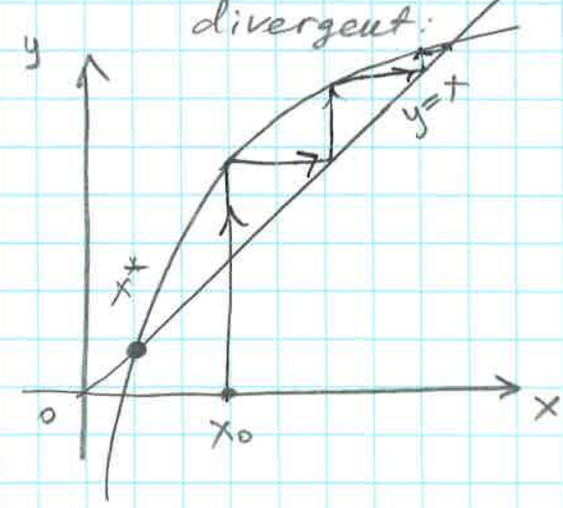
(does not fit any fixed pt problem)

Convergence of $x_{k+1} = \varphi(x_k)$:

convergent:



divergent:



Q: How do we know if the fixed-pt iteration will converge or not?

Theorem (4.5.1)

Let $\varphi \in C^1$ and $|\varphi'(x)| < 1$ in some interval $[x^* - \delta, x^* + \delta]$ centered about a fixed pt. x^* of φ.

If $x_0 \in [x^* - \delta, x^* + \delta]$ then the fixed point iteration converges to x^*.

Proof:

$$x_{k+1} = \varphi(x_k) \overset{\text{Taylor's}}{\underset{\text{about } x^*}{=}} \varphi(x^*) + \varphi'(\underbrace{\xi_k}_{\substack{\text{between} \\ x_k \ \& \ x^*}})(x_k - x^*)$$

$$\Rightarrow x_{k+1} - x^* = \underbrace{\varphi(x^*)}_{x^*} - x^* + \varphi'(\xi_k)(x_k - x^*)$$

$$\Rightarrow \underbrace{x_{k+1} - x^*}_{e_{k+1}} = \varphi'(\xi_k)\underbrace{(x_k - x^*)}_{e_k}$$

$$\Rightarrow |e_{k+1}| = |e_k| \underbrace{|\varphi'(\xi_k)|}_{<1}, \text{ i.e.}$$

if $x_0 \in [x^* - \delta, x^* + \delta]$

$$\lim_{k \to \infty} \frac{|e_{k+1}|}{|e_k|} = \lim_{k \to \infty} |\varphi'(\xi_k)|$$

$$= \underbrace{|\varphi'(x^*)|}_{\text{limit of the error reduction factor.}}$$

$\square$

Note: convergence is guaranteed if φ is a contraction (map),

i.e.: there exists $0 \leq L < 1$ s.t.

$$|\varphi(x) - \varphi(y)| \leq L\,|x - y| \quad \forall x, y.$$

(the smallest L is called the Lipschitz constant)

Theorem (4.52)

If φ is a contraction, then it has a unique fixed point x^* and the iteration $x_{k+1} = \varphi(x_k)$, $k=0,1,2,\ldots$, converges to x^* from any x_0.

Proof: p.97. Shows that $\{x_k\}_{k=0}^{\infty}$ is a Cauchy sequence $\Rightarrow$ converges

$\forall \varepsilon > 0 \ \exists N$ s.t. if $m,n > N \Rightarrow |x_m - x_n| < \varepsilon$

More Examples: Let $x_0 = 0$ and

consider $\boxed{f(x) = x^3 + 6x^2 - 8 = 0}$ (root finding)

(1) $x = \varphi(x) = f(x) + x = x^3 + 6x^2 + x - 8$ (fixed pt prob)

$\varphi'(x) = 3x^2 + 12x + 1 > 1 \quad \forall x > 0 \Rightarrow$ divergence!

(For instance, if $x = 1.5 \Rightarrow x_{k+1} = \varphi(x_k)$ diverges)

(Note: $f(x) = 0$ has solution on $[1,2]$ by the IVT: $f(1) = -1 < 0$ & $f(2) = 24 > 0$!)

(2) for the same $f(x) = 0$, now let $\varphi(x) = \sqrt{\frac{8}{x+6}}$

How so?

$$x^3 + 6x^2 - 8 = 0$$
$$x^3 + 6x^2 = 8$$
$$x^2(x+6) = 8$$
$$x^2 = \frac{8}{x+6} \Rightarrow \text{for } x > 0,\ x = \underbrace{\sqrt{\frac{8}{x+6}}}_{\varphi(x)}$$

$$|\varphi'(x)| = \frac{\sqrt{2}}{(x+6)^{3/2}} < 1 \quad \text{if} \quad (x+6)^{3/2} > \sqrt{2}$$

want for convergence

$x+6 > (\sqrt{2})^{2/3}$

$x > 2^{1/3} - 6 \approx -4.74$

Since $x^* \in [1,2]$, for $x_0 = 1.5$, the distance

$|x_0 - x^*| \le 0.5 \Rightarrow$ if $\delta = 0.5 \Rightarrow$

$[x^* - 0.5, x^* + 0.5]$ contains $x_0 = 1.5$ and

since $|\varphi'(x)| < 1$ there $\Rightarrow$ convergence is guaranteed!

(3) for the same $f(x)=0$, now let $\varphi(x) = \sqrt{\frac{8-x^3}{6}}$.

(Note: $x^3 + 6x^2 - 8 = 0$

$6x^2 = 8 - x^3$

$x = \sqrt{\frac{8-x^3}{6}}, \quad 0 < x (\le 2)$)

So, $f(x) = 0 \iff x = \varphi(x) = \sqrt{\frac{8-x^3}{6}}$

$$\varphi'(x) = \frac{1}{2}\left(\frac{8-x^3}{6}\right)^{-1/2}\left(-\frac{3}{6}x^2\right) = -\frac{x^2}{4}\sqrt{\frac{6}{8-x^3}}$$

and $|\varphi'(x)| = \frac{x^2}{4}\sqrt{\frac{6}{8-x^3}} < 1$ when $\underline{x < 1.6}$ (show!)

If $x_0 = 1.5 \Rightarrow$ then like in (2), $|\varphi'(x)| < 1$ on $[x^* - 0.5, x^* + 0.5]$. (If $x_0 \in [1,2]$, but $x > 1.6$, we could not conclude if the fixed pt iteration will conclude. The interval $[x^* - 0.5, x^* + 0.5]$ is taken since one could see from (2) that $x^* = 1.06$)

Notes: 1) Note that $[x^*-\delta, x^*+\delta]$ is centered about x^* to guarantee convergence.

2) if φ is a contraction, there is no need in existence of of $\varphi'(x)$ (and $|\varphi'(x)|<1$)

Example:

Consider $f(x) = x^2-x-1=0$, $x>1$.

$$x^* = \frac{1+\sqrt{5}}{2} \quad \text{(golden ratio!)}$$

If $\varphi(x) = 1+\frac{1}{x} \Rightarrow x = 1+\frac{1}{x}$ is equivalent to $f(x)=0$ (check!)

Is φ a contraction?

$$|\varphi(x)-\varphi(y)| = |\cancel{1}+\frac{1}{x}-\cancel{1}-\frac{1}{y}|$$

$$= |(x-y)/xy| = |x-y| / \underbrace{|xy|}_{?}$$

If $x,y>1 \Rightarrow L = \frac{1}{|xy|} < 1$

So, $\varphi(x) = 1+\frac{1}{x}$ is a contraction map for $x>1$

$\Rightarrow \forall x_0>1 \quad x_{k+1} = \varphi(x_k)$ converges to x^*.
$k=0,1,2,\dots$

Note: $x_1^* = \frac{1+\sqrt{5}}{2} \approx 1.618$, $x_2^* = \frac{1-\sqrt{5}}{2} \approx -0.618$ are both fixed pts of $\varphi(x) = 1+\frac{1}{x}$.

x_1^* is an attractor! ($|\varphi'(x_1^*)| \approx 0.38 < 1$), while

x_2^* is unstable ($|\varphi'(x_2^*)| \approx 2.61 > 1$).