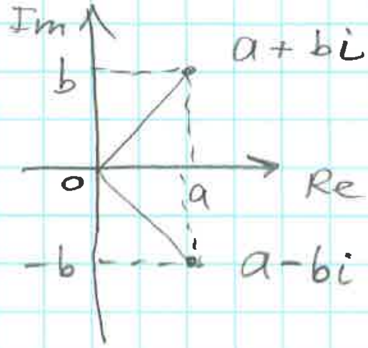
§4.6 Fractals: Julia & Mandelbrot Sets.

Complex plane:



$\sqrt{-1} = i$

$z = a + bi$

$\bar{z} = a - bi$

$|z| = \sqrt{a^2+b^2}$

modulus or abs. value of z

$\mathbb{C}$ - set of complex numbers

Euler's formula:

$e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow$ for any $z \in \mathbb{C}$, $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

with $r = |z|$ $\quad \left(\bar{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta) \right)$

We will consider fixed point iteration and Newton's Method (briefly) in $\mathbb{C}$.

fixed pts are $0, 1$.

Example: take $\varphi(z) = z^2$, then fixed pt. iteration is $z_{k+1} = z_k^2$, $k = 0, 1, 2, \ldots$, given $z_0 \in \mathbb{C}$.

If $|z_0| < 1 \Rightarrow \{z_k\}$ converges to $z^* = 0$.

If $|z_0| > 1 \Rightarrow \{z_k\}$ grows in modulus $\Rightarrow$ diverges from 0 and 1.

If $|z_0| = 1 \Rightarrow |z_k| = 1 \ \forall k$. For $z_0 = 1$ or $z_0 = -1$ then $\{z_k\}$ settles down to $z^* = 1$ right away.

If $|z_0| = 1$, $z_0 \neq \pm 1$, say $z_0 = e^{2\pi i/3}$

$= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ then $z_1 = e^{4\pi i/3}$

$= \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, $\quad z_2 = e^{8\pi i/3} = e^{2\pi i/3} = z_0$!

$(8\pi/3 = 2\pi + 2\pi/3)$

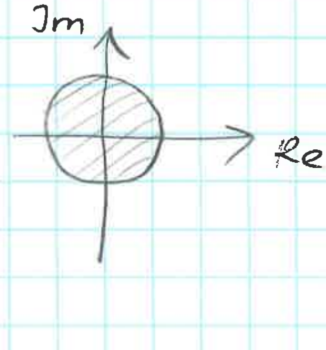
$\Rightarrow$ and the cycle $z_0 \to z_1 \to z_2 = z_0$ repeats! Another possibility: $z_0 = e^{2\pi i \alpha}$, where α is irrational $\left(z_0 = e^{2\sqrt{2}\pi i} \right) \Rightarrow$ no repetition as k grows, but $\{z_k\}$ becomes dense on the unit circle.

$$z_0 = e^{2\sqrt{2}\pi i},\quad z_1 = e^{4\sqrt{2}\pi i},\quad z_2 = e^{8\sqrt{2}\pi i},\quad z_3 = e^{16\sqrt{2}\pi i},\ \ldots$$

The sequence $z_0,\ z_1 = \varphi(z_0),\ z_2 = \varphi(z_1) = \varphi(\varphi(z_0)), \ldots$ forms a sequence called the <u>orbit of z_0 under φ.</u>

For a polynomial function φ, the set of all start points z_0, for which the orbit remains bounded, is called the filled Julia set for φ, and its boundary — the Julia set (Gaston Julia, 1893-1978)

<u>Ex</u>: For $\varphi(z) = z^2$, the filled Julia set is the closed unit disk with the unit circle being the Julia set.



- Consider:

$$\varphi(z) = z^2 + c \quad \text{with}$$

$$c = -1.25$$

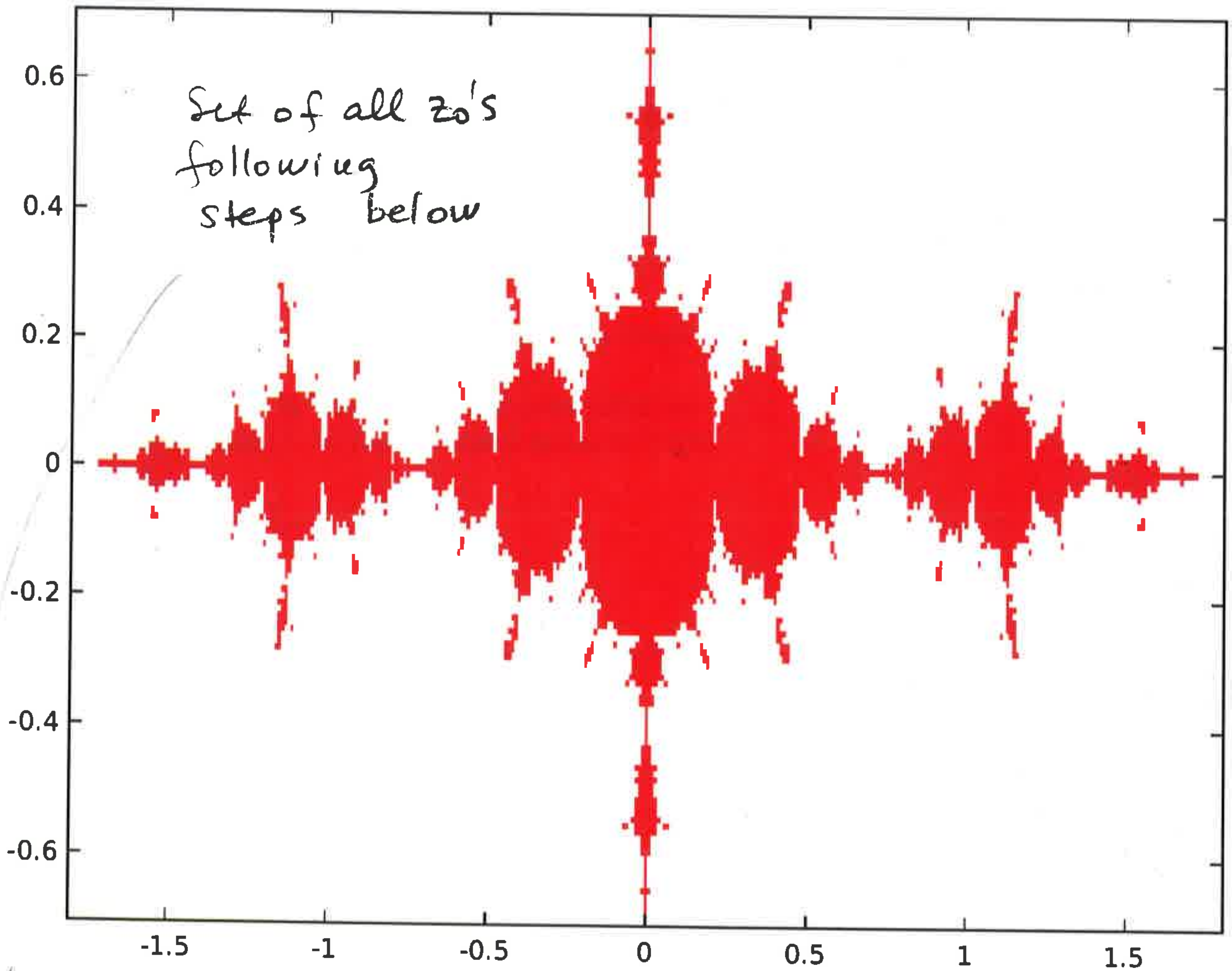
(see MATLAB code JuliaSet.m on the textbook web page.)

Julia Set (filled) for $\varphi(z) = z^2 - 1.25$ ③

The boundary is a fractal

(fractals are self-similar and have fractional dimension)

$z_1^* = \frac{1+\sqrt{6}}{2}, \quad z_2^* = \frac{1-\sqrt{6}}{2}$ → fixed points



To get the plot: Start w/ many different z_0's in a rectangular region than contains the two fixed pts of φ, and run the fixed point iteration to get one of the 3 possibilities:

1) $|z_k| \geq 2 \Rightarrow$ unbounded orbit, we don't use this z_0
2) $|z_k - z^*| \leq 10^{-6}$ and it's true for 5 iterations in a row $\Rightarrow$ include z_0! OR:
3) # iterations $k \geq 100$ & $|z_k|$ bounded $\Rightarrow$ include z_0

(Note: if $|z_k| \geq 2 \Rightarrow |z_{k+1}| = |z_k^2 - 1.25|$

$\geq |z_k^2| - 1.25 = |z_k|^2 - 1.25 \geq 4 - 1.25 = 2.75$

$\Rightarrow |z_{k+2}| \geq |z_{k+1}|^2 - 1.25 = 2.75^2 - 1.25 \approx 6.3, \ldots$

So $\{z_k\}_{k=1}^n$ is unbounded in this case.)

Could try: different # iterations (50 or 200)

Fact: for $\varphi(z) = z^2 + c$, Julia sets may be connected, i.e., if one can move between any 2 pts in the set without leaving the set.

To be connected, Julia set has to have $z_0 = 0$ in it.

Benoit Mandelbrot (1982) studied c-values that generate connected Julia sets, using $z_0 = 0$ and fixed pt iteration. So, this set of c values is called the Mandelbrot set.

So, the Mandelbrot set is the set of all c values (in $\varphi(z) = z^2 + c$) for which the orbit of 0 under φ, i.e. $\{0, \varphi(0), \varphi(\varphi(0)), \varphi(\varphi(\varphi(0))), \ldots\}$ remains bounded.

Example: $z_{k+1} = z_k^2 + c$, $z_0 = 0$, $|z_k| \leq 2$?
$k = 0, 1, 2$

If $c = 1 \Rightarrow z_0 = 1, z_1 = 1, z_2 = 2, z_3 = 5 \Rightarrow$ grows $\Rightarrow c = 1$ is not in the set

If $c = -1 \Rightarrow z_0 = 0, z_1 = 1, z_2 = 0, z_3 = -1 \Rightarrow$ bounded, so $c = -1$ is in the set.

- We can apply Newton's Method to problems in the complex plane $\mathbb{C}$:

Looking for roots of $p(z)$ w/ N.M. gives iteration $z_{k+1} = \underbrace{z_k - \frac{p(z_k)}{p'(z_k)}}_{\varphi(z_k)}$, w/ z_0.

See example 4.16, p.102.

$p(z) = z^3 + 1 = 0$ (roots are $z = -1$, $z = e^{\pi i/3}$, $z = {}^{-\pi i/3}$)