§8.6.1. Piecewise Cubic Hermite Interpolation.

Consider piecewice cubic p to match f, f' at the nodes $x_0, x_1, \dots, x_n$ with $h = x_i - x_{i-1}$ $\forall i$.

p - cubic $\Rightarrow$ p' - quadratic, and

$$p(x_i) = f_i, \quad p'(x_i) = f_i'.$$

$$\text{So, } p'(x) = f'_{i-1}\frac{x-x_i}{\underbrace{x_{i-1}-x_i}_{=-h}} + f'_i \frac{x-x_{i-1}}{\underbrace{x_i - x_{i-1}}_{=h}} + \alpha\underbrace{(x-x_{i-1})(x-x_i)}_{\text{quadratic part}}$$

on $[x_{i-1}, x_i]$

where α - additional parameter that can be varied to match f values.

$$\Rightarrow p(x) = \int_{x_{i-1}}^{x} p'(t)\,dt + C = -\frac{f_i'}{h}\int_{x_{i-1}}^{x}(t-x_i)\,dt$$

$$+ \frac{f_i'}{h}\int_{x_{i-1}}^{x}(t-x_{i-1})\,dt + \alpha\int_{x_{i-1}}^{x}(t-x_{i-1})(t-x_i)\,dt + C$$

From $p(x_{i-1}) = f_{i-1} \Rightarrow C = f_{i-1} \Rightarrow$ on $[x_{i-1}, x_i]$

$$p(x) = -\frac{f'_{i-1}}{h}\left(\frac{(x-x_i)^2}{2} - \frac{h^2}{2}\right) + \frac{f'_i}{h}\frac{(x-x_{i-1})^2}{2}$$

$$+ \alpha(x-x_{i-1})^2\left(\frac{x-x_{i-1}}{3} - \frac{h}{2}\right) + f_{i-1}$$

Since we also need $p(x_i) = f_{i-1}$ $\Rightarrow$

$$p(x_i) = f'_{i-1}\frac{h}{2} + f'_i\frac{h}{2} - \alpha\frac{h^3}{6} + f_{i-1} = f_i \Rightarrow$$

$$\alpha = \frac{3}{h^2}\left(f'_{i-1} + f'_i\right) + \frac{6}{h^3}\left(f_{i-1} - f_i\right).$$

<u>Example</u>: Given $f(x) = x^4$ on $[0, 2]$, find the piecewise cubic Hermite interpolant

of f using subintervals $[0,1]$, $[1,2]$.

So, we have: $x_0 = 0$, $x_1 = 1$, $x_2 = 2$
$f_0 = 0$, $f_1 = 1$, $f_2 = 16$ and $h = 1$.
$f_0' = 0$, $f_1' = 4$, $f_2' = 32$

on $[0,1]$:

$$\Rightarrow p(x) = -\frac{0}{1}\left(\frac{(x-1)^2}{2} - \frac{1^2}{2}\right) + \frac{4}{1}\frac{(x-0)^2}{2} + \alpha(x-0)^2\left(\frac{x-0}{3} - \frac{1}{2}\right) + 0$$

$$= 2x^2 + \alpha x^2\left(\frac{x}{3} - \frac{1}{2}\right)$$

α? $\alpha = \frac{3}{1^2}(0+4) + \frac{6}{1^3}(0-1) = 6 \Rightarrow$

$$p(x) = 2x^2 + 6x^2\left(\frac{x}{3} - \frac{1}{2}\right) = 2x^3 - x^2 \quad \text{on } [0,1]$$

Similarly, one can show that

$$p(x) = 6x^3 - 13x^2 + 12x - 4 \quad \text{on } [1,2]$$

Thus, $p(x) = \begin{cases} 2x^3 - x^2, & x \in [0,1] \\ 6x^3 - 13x^2 + 12x - 4, & x \in [1,2] \end{cases}$

(Check if p and f, p' and f' match at 0, 1, 2.)