

## ① Chapter 11. Numerical Solution of the IVP for ODES.

IVP:  $y'(t) = f(t, y(t))$  ODE

$$y(t_0) = y_0$$

$$t \geq t_0$$

IC  
(initial condition)

### Examples

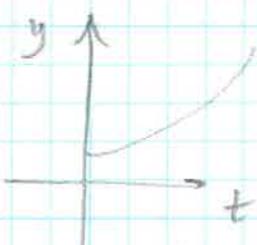
① Malthusian Model (Thomas Malthus, 1798)

$y(t)$  is population at time  $t$

$\underbrace{y' \approx y}_{\text{rate of growth of } y} \Rightarrow \underbrace{y' = ky}_{\text{(for some constant } k)}$  ODE can be solved analytically:

$$\frac{dy}{dt} = ky \Rightarrow \frac{dy}{y} = k dt \Rightarrow \int \frac{dy}{y} = \int k dt$$

$$\Rightarrow \ln y = kt + C_1 \Rightarrow y = e^{\underbrace{kt+C_1}_C} = Ce^{kt}$$



for  $k > 0$ , population grows exponentially!

If  $y(0) = 100 \Rightarrow C = 100 \Rightarrow y = 100e^{kt}$   
( $k$  is the birth rate, called Malthusian parameter.)

② Often analytic formulas for solutions of IVP do not exist or difficult to find:

Consider Lotka-Volterra predator-prey  
(1925) (1926) equations:

Let  $F(t) = \#$  foxes (predators) at time  $t$       ②  
 $R(t) = \#$  rabbits (prey) at time  $t$   
in a closed environment. Then

$$\left. \begin{aligned} \frac{dR}{dt} &= (\alpha - \beta F) R \\ \frac{dF}{dt} &= (\gamma - \delta R) F \end{aligned} \right\} \begin{matrix} \alpha, \beta, \gamma, \delta \text{ are} \\ \text{positive parameters} \end{matrix}$$

(2R)

- Rate of growth of rabbit population increases with increasing  $R(t)$  but decreases with increasing  $F(t)$  (foxes eat rabbits).      (- $\beta F$ )
- Rate of growth of fox population decreases without rabbits (no food), but increases if foxes eat rabbits (+ $\gamma R F$ ).      ( $-\delta R$ )
- The system cannot be solved analytically, but we can find a numerical solution given initial conditions for  $R$  &  $F$ .

## S 11.1 Existence & Uniqueness of Solutions.

(1)

$$(*) \quad \text{IVP} \quad \begin{cases} y'(t) = f(t, y(t)), \quad t \geq t_0 \\ y(t_0) = y_0 \end{cases}$$

Examples : (1)  $y' = y \tan t$   
 $y(0) = 1$

$\Rightarrow$  solution  $y(t) = \sec t$  DNE when  
 $t = \pm \frac{\pi}{2}$

$\Rightarrow$  so,  $y(t) = \sec(t)$  exists  
locally.

$$\Rightarrow -\frac{\pi}{2} < t < \frac{\pi}{2}$$

( $\sec \rightarrow \pm \infty$   
 $\rightarrow \pm \frac{\pi}{2}$ )

(2)  $y' = y^2$   
 $y(0) = 1$   $\Rightarrow$  sol. is  $y(t) = \frac{1}{1-t}$  DNE when  
 $t=1$   
 $\Rightarrow$  so,  $y(t) = \frac{1}{1-t}$  for  $t \in [0, 1)$ , no sol. beyond  
 $t=1$ .

The following thm. gives sufficient conditions  
for the IVP to have a solution locally:

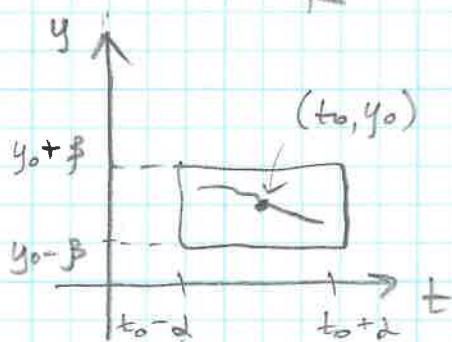
Thm. (11.1.1) If  $f(t, y)$  is continuous in a  
func. of 2 var's

a rectangle  $R$  centered about  $(t_0, y_0)$  :

$R = \{(t, y) \mid |t - t_0| \leq d, |y - y_0| \leq \beta\}$  then the IVP

(\*) has a solution  $y(t)$  for  $|t - t_0| \leq \min(d, \frac{\beta}{M})$   
where  $M = \max_R |f(t, y)|$ .

$\exists \varepsilon > 0$



Even if a sol. exists,  
it may not be unique:

IVP  $y' = y^{2/3}$ ,  $y(0) = 0$  has  
solutions  $y(t) = 0$  &  $y(t) = \frac{t^3}{27}$

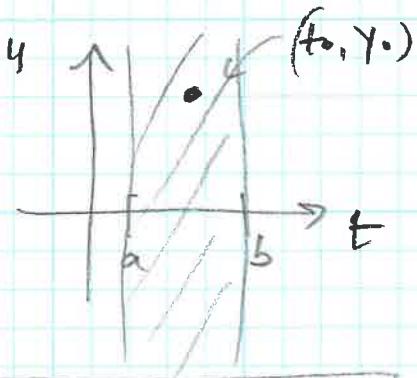
Thm (11.1.2) If both  $f$  &  $\frac{\partial f}{\partial y}$  are cont. (2)

in the rectangle  $R$  (see above) then the IVP  $\star$  has a unique sol. for  $|t-t_0| \leq \min(a, \frac{B}{M})$  where  $M = \max_R |f(t,y)|$ .

In the example  $y' = \underbrace{y^{2/3}}_{f(t,y)}, y(0)=0$ ,  $\frac{\partial f}{\partial y} = \frac{2}{3y^{1/3}}$  is not continuous in any rectangle about  $(0,0)$ .

- Global Existence & Uniqueness.

Thm (11.1.3) Assume that  $t_0 \in [a, b]$ . If  $f$  is continuous in the strip  $a \leq t \leq b, -\infty < y < \infty$  and uniformly Lipschitz continuous in  $y$  — i.e.,  $\exists L$  s.t.  $\forall y_1, y_2$  and  $t \in [a, b]$ ,  $|f(t, y_2) - f(t, y_1)| \leq L |y_2 - y_1|$  — then IVP  $\star$  has a unique sol. in  $[a, b]$ .



Note: If  $\frac{\partial f}{\partial y}$  exists the  $f$  satisfies the Lipschitz condition if  $\exists L$  s.t.  $|\frac{\partial f}{\partial y}| \leq L$ . Why?

Recall:

$$f(t, y_2) = \underset{\text{Taylor's}}{f(t, y_1) + (y_2 - y_1) \frac{\partial f}{\partial y}(t, \bar{y})} \quad (\text{if } y_1 < \bar{y} < y_2 \\ (\text{if } y_1 < y_2))$$

$$\Rightarrow |f(t, y_2) - f(t, y_1)| \leq \max \left| \frac{\partial f}{\partial y} \right| |y_2 - y_1|$$

that is,  $f$  is Lipschitz cont. in this case.

(Note: contraction map ( $L < 1$ ) is a special type of Lipschitz continuity)

One more theorem is

Thm (11.1.4) Suppose  $y(t)$  and  $z(t)$  satisfy

$$\begin{aligned} y' &= f(t, y) \quad \text{and} \quad z' = f(t, z), \\ y(t_0) &= y_0 \end{aligned}$$

small  
small change in data!

Then, if  $f$  is continuous  $\forall t \in [a, b]$  and  
 $\forall y$  and uniformly Lipschitz continuous in  $y$ ,  
then, for  $\forall t \in [a, b]$ ,  $|z(t) - y(t)| \leq e^{\frac{L|t-t_0|}{\delta_0}} |y_0|$ .

\* That is, small change in data ( $y_0$ ) results  
in small change in solution  $\Rightarrow$  we say,  
the IVP is well-posed. Good property in NA.  
since we deal w/ rounding!