§ 8.4 The Error in Polynomial Interpolation.

Data: $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n) \Rightarrow$ (w/ $p(x_i) = y_i$)

Let p be the interpolation polynomial of deg n.

If $f(x)$ is a function w/ $\underset{i=0,1,\dots n}{f(x_i) = y_i}$ that we interpolate w/ $p(x)$, how large is the difference between $f(x)$ & $p(x)$ at points other than x_i's ?

Theorem (8.4.1) Assume that $f \in C^{n+1}[a,b]$ and $x_0, \dots, x_n$ are in $[a,b]$. Let $p(x)$ be a polynomial of deg n that interpolates f at $x_0, \dots, x_n$. Then, for any x in $[a,b]$,

$$\boxed{f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{j=0}^{n} (x - x_j)}$$

for some ξ_x in $[a,b]$.

(We will prove this later; also, see p. 190)

[*Note:* $f[\underbrace{x_0, x_1, \dots, x_n}_{\text{data}}, \underset{\substack{\uparrow \\ \text{new point}}}{x}] = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)$]

Example (p. 191) Let $f(x) = \sin x$

If we use p_1 of deg. 1 to interpolate $\sin x$ using $x_0 = 0$ and $x_1 = \frac{\pi}{2}$, then interval is $[0, \frac{\pi}{2}]$

and $p_1(0) = \underbrace{\sin 0}_{f(0)} = 0$ and $p_1(\pi/2) = \underbrace{\sin \pi/2}_{f(\pi/2)} = 1$

$p_1(x) = a_0 + a_1(x-0) = a_0 + a_1 x$

$p_1(0) = a_0 = 0$, $p_1(\pi/2) = a_1 \pi/2 = 1 \Rightarrow a_1 = 2/\pi$

So, $p_1(x) = \frac{2}{\pi} x$

The Thm. says:

$$|f(x) - p_1(x)| = \Big|\frac{1}{2!} f''(\underbrace{\xi_x}_{\text{on } [0,\pi/2]})(x-0)(x-\frac{\pi}{2})\Big| \leq$$

$$\left(|f''(x)| = |-\sin x| \leq 1 \right)$$

$$\leq \Big|\frac{1}{2} x(x-\frac{\pi}{2})\Big| \leq \Big|\frac{1}{2}(-\frac{\pi^2}{16})\Big| = \frac{\pi^2}{32} \approx 0.3084$$

$x^2 - \frac{\pi}{2}x$ has max. at $x = \pi/4$, $\left(\frac{\pi}{4}\right)^2 - \frac{\pi}{2} \cdot \frac{\pi}{4} = -\frac{\pi^2}{16}$

So, $|\underbrace{\sin x}_{f} - \underbrace{\frac{2}{\pi} x}_{p_1}| \leq 0.3084$

Q: Suppose we have $n+1$ nodes $x_0, \dots, x_n$ on $[0, \pi/2]$.
Will $p_n(x)$ be able to extrapolate, say, $x = \pi$
w/ reasonable error $|f(\pi) - p_n(\pi)|$?

Since $\sin x$ has only bounded derivatives
and since $\frac{1}{(n+1)!}$ decreases as $n \nearrow$ and the
factor $\left|\prod_{j=0}^{n} (\pi - x_j)\right| = |(\pi - x_0)(\pi - x_1) \dots (\pi - x_n)| \leq \pi^{n+1}$,
$x_i \in [0, \pi/2]$
then:

$|\text{error}| \leq \frac{\pi^{n+1}}{(n+1)!} \xrightarrow[n\to\infty]{} 0$ since $(n+1)! \nearrow$ faster than π^{n+1}.

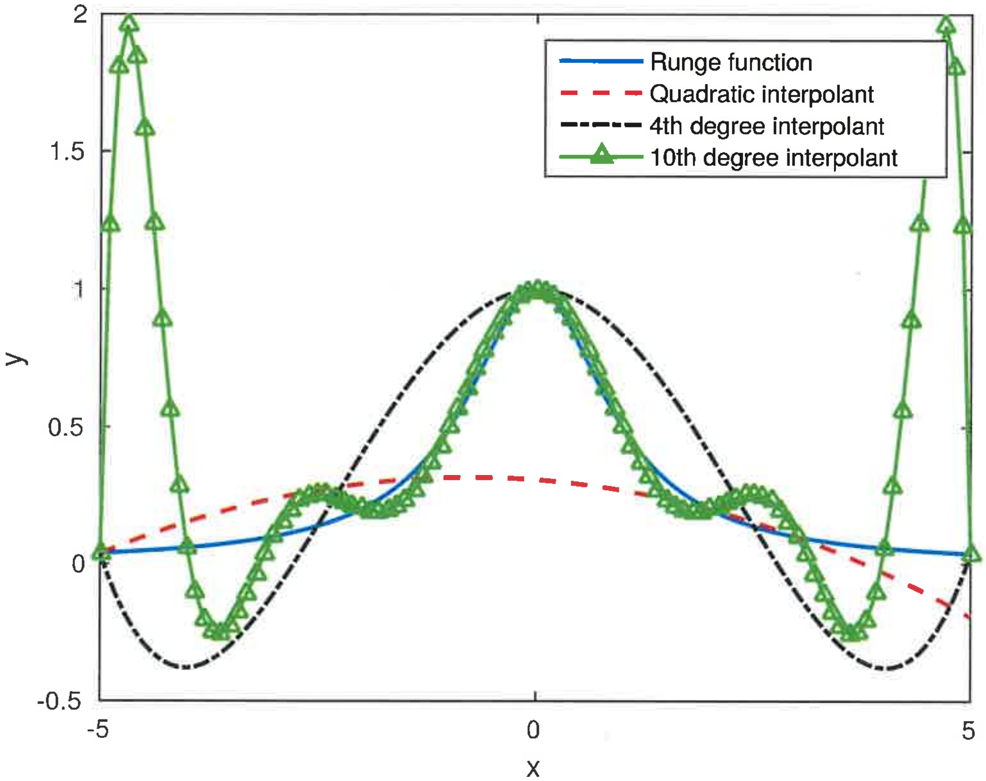
$\Rightarrow$ looks good.

Runge phenomenon

$$f(x) = \frac{1}{1+x^2} \quad \text{on } [-5,5]$$

$$x_i = -5 + \frac{10i}{n}, \quad i = 0,1,\dots,n$$

(equally spaced nodes)



In the following example (#8.4.2),
consider $f(x) = \underbrace{\frac{1}{1+x^2}}_{\text{the Runge function}}$ on $[-5,5]$ (HW3 → 4(a))

If f is interpolated by polynomials of deg n, with more and more $\underbrace{\text{nodes}}_{\substack{\text{equally spaced} \\ \text{in } [-5,5]}}$ ($n \nearrow$)
then oscillations of graphs of polynomials near the ends of $[-5,5]$ become larger and larger. Why?

By thm. 8.4.1,

$$\left| \frac{1}{1+x^2} - p_n(x) \right| = \left| \frac{1}{(n+1)!} f^{(n+1)}(\underbrace{\xi_x}_{\text{in } [-5,5]}) \prod_{j=0}^{n} (x - x_j) \right|$$

w/ $f^{(n+1)}(\xi_x) \prod_{j=0}^{n} (x - x_j)$ growing faster with n than $(n+1)!$ near ± 5,
that is, the error $\to \infty$ as $n \to \infty$.

(Nodes in $[-5,5]$: $x_i = -5 + \frac{10i}{n}$, $i = 0, 1, \ldots, n$)

This problem is called <u>Runge's phenomenon</u>

Now, back to the <u>proof of thm. 8.4.1</u> :

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Proof of Thm. 8.4.1

Let $x_0, x_1, \ldots, x_n$ be nodes in $[a,b]$, and let $x \in [a,b]$.

Consider $q(t) = p(t) + \lambda \prod_{j=0}^{n} (t - x_j)$ with $\xrightarrow{} f(x_i) = p(x_i)$

$$\lambda = \frac{f(x) - p(x)}{\prod_{j=0}^{n} (x - x_j)}.$$

Note: degree of q is $n+1$

and q interpolates f at $\underbrace{x_0, \ldots, x_n, x}_{n+2 \text{ nodes}}$ in $[a,b]$.

Let $s(t) \equiv f(t) - q(t) \Rightarrow$ continuous on $[a,b]$,

$s(x) = f(x) - q(x) = 0$, $s(x_j) = 0$ $(j = 0, 1, \ldots, n)$

as well. So, by Rolle's thm, $s' = 0$ somewhere $(n+1)$ times between pts $\underbrace{x_0, x_1, \ldots, x_n, x}_{n+2 \text{ pts}}$. If we apply Rolle's thm again (to s') $\Rightarrow$ $s'' = 0$ somewhere at n points, ..., etc., till $s^{(n+1)} = 0$ at one pt in $[a,b]$. Let's call this pt. ξ_x. Thus,

$$s^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - q^{(n+1)}(\xi_x) = 0$$

If we differentiate q $(n+1)$ times, we get.

$$q^{(n+1)}(t) = \underbrace{p^{(n+1)}(t)}_{\overset{\shortparallel}{0}} + \lambda (n+1)! = f^{(n+1)}(\xi_x)$$

$(\deg p = n)$

$$\Rightarrow \lambda (n+1)! = \underbrace{\frac{f(x) - p(x)}{\prod_{j=0}^{n} (x - x_j)}}_{\lambda} (n+1)! = f^{(n+1)}(\xi_x)$$

$$\Rightarrow f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{j=0}^{n} (x - x_j) \quad \square.$$