Chapter 8. Polynomial Interpolation.

§§ 8.1, 8.2 The Lagrange Form of Polynomial Interpolation.

Consider data:

$$(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),\ \dots,\ (x_n, y_n)$$

Goal: find a polynomial of degree at most n to fit the data, $y = p(x) = \sum_{k=0}^{n} c_k x^k$

One way to do it (Chapter 7):

Since we want $y_i = c_0 + c_1 x_i + c_2 x_i^2 + \dots + c_n x_i^n$
$i = 0, 1, 2, \dots, n$

then the coeff's c_k must satisfy:

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & \dots & x_2^n \\ & & \dots & & \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{pmatrix}}_{\text{Vandermonde matrix}} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$\Rightarrow$ solve by Gaussian elimination

* Finding such a polynomial p gives us a way to estimate points between known data points; p will be an interpolating polynomial, given pts (x_i, y_i) are interpolation points (or nodes).

To find p for given data, we need

$$p(x_i) = y_i \quad , \quad i = 0, 1, \ldots, n.$$

$$\deg p \leq n$$

Let us start with $\varphi_i(x)$:

$$\varphi_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_i - x_0} \cdot \frac{x - x_1}{x_i - x_1} \cdot \ldots \frac{x - x_{i-1}}{x_i - x_{i-1}} \frac{x - x_{i+1}}{x_i - x_{i+1}} \ldots \frac{x - x_n}{x_i - x_n}$$

Note $\varphi_i(x_i) = 1$ and $\underset{j \neq i}{\varphi_i(x_j) = 0}$

$\varphi_i(x)$ has n factors (all but one of $i = 0, 1, \ldots, n$)

Now consider $y_i \varphi_i(x)$ with

$$y_i \varphi_i(x_i) = y_i \quad \text{and} \quad y_i \varphi_i(x_j) = 0 \quad (j \neq i)$$

Polynomial of (at most) deg. n that interpolates the points (x_i, y_i), $i = 0, 1, \ldots, n$,

is

$$\underbrace{p(x) = \sum_{i=0}^{n} y_i \varphi_i(x) = \sum_{i=0}^{n} y_i \left(\prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \right)}_{\text{called the Lagrange form of } p.} \quad (*)$$

Example: Fit a quadratic polynomial to the data pts $(1,2)$, $(2,3)$, $(3,6)$.

Lagrange form gives

$$p(x) = y_0 \underbrace{\frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2}}_{\varphi_0(x)} + y_1 \underbrace{\frac{x - x_0}{x_1 - x_0} \frac{x - x_2}{x_1 - x_2}}_{\varphi_1(x)} + y_2 \underbrace{\frac{x - x_0}{x_2 - x_0} \frac{x - x_1}{x_2 - x_1}}_{\varphi_2(x)}$$

$$= 2\,\frac{x-2}{1-2}\cdot\frac{x-3}{1-3} + 3\,\frac{x-1}{2-1}\,\frac{x-3}{2-3} + 6\,\frac{x-1}{3-1}\,\frac{x-2}{3-2}$$

$$= \dots = x^2-2x+3$$

$p(x_0)=p(1) = 2 = y_0$; $p(x_1)=p(2)=3=y_1$; $p(x_2)=p(3)=6=y_2$

$p(x)$ is simply written down, little computational time is involved. If we want to evaluate p at a new point x, it will require:

$$p(x) = \underset{\substack{\downarrow \\ n \text{ additions}}}{\sum_{i=0}^{n}} y_i \left(\underbrace{\prod_{j\neq i} \frac{x-x_j}{x_i-x_j}}_{(n+1) \text{ multiplications}} \right) \quad \times$$

$\Rightarrow \quad n(n+1) = n^2+n$ alg oper's

$\Rightarrow \quad O(n^2)$ alg. operations

Theorem (8.2.1)

If the nodes $x_0, x_1, \dots, x_n$ are distinct then for any values $y_0, y_1, \dots, y_n$ there exists a unique polynomial p of degree $\leq n$ s.t. $p(x_i) = y_i$, $i = 0, 1, \dots, n$.

Proof: Recall $\underbrace{p(x) = \sum_{i=0}^{n} y_i\, \varphi_i(x)}_{\text{Lagrange form}}$ w/

$p(x_i)=y_i$ by construction.

Why is this unique? If we had another

polynomial q ($\deg q \le n$) s.t.

$q(x_i) = p(x_i) = y_i$, $i = 0, 1, \dots, n$. Then

$(p-q)(x_i) = 0$, $i = 0, 1, \dots, n \Rightarrow$ by the algebra fact that if a polynomial of $\deg \le n$ vanishes at $n+1$ distinct pts, it must be identically 0, we have $p - q \equiv 0$ and $p \equiv q$ $\square$.

Note: # operations (again)

if $y_i = \sum_{j=0}^{n} c_j x_i^j$, $i = 0, \dots, n$ then the value of $p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n$ at x can be determined using Horner's rule w/ $O(n)$ operations:

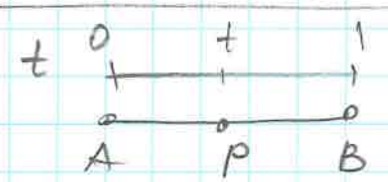
Set $y = c_n$ then $y \to yx + c_{n-1} = c_n x + c_{n-1}$,

next $y \to yx + c_{n-2} = c_n x^2 + c_{n-1} x + c_{n-2}$, ...

after n steps, $y \to c_0 + c_1 x + \dots + c_{n-1} x^{n-1} + c_n x^n$.

Each of n steps has 2 operations $\Rightarrow 2n \Rightarrow O(n)$.

Modified Lagrange form: using barycentric coordinates.



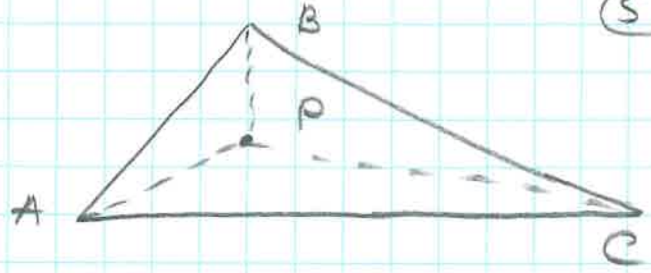
for AB, and any P between them,

$P = (1-t)A + tB$, $t \in [0,1]$

Note if $\alpha = 1-t$, $\beta = t$ then $\alpha + \beta = 1$

Now: 3 points A, B, C

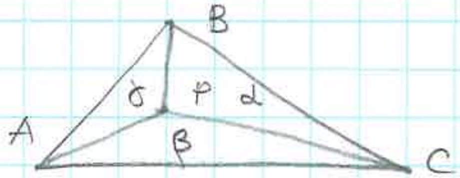
& P inside ΔABC
(can be on one of the sides too)



α, β, γ are the barycentric coord's of P

if $\alpha + \beta + \gamma = 1$ & $P = \alpha A + \beta B + \gamma C$. → convex combination

P is a "weighted average" of A, B, C.



$\alpha = \dfrac{BPC}{ABC}$, $\beta = \dfrac{APC}{ABC}$, $\gamma = \dfrac{APB}{ABC}$

We will use this idea of "weighted average" for interpolation: interpolating polynomial will be a "weighted average" of function values.

(1) Define $\varphi(x) = \prod_{j=0}^{n} (x - x_j) = (x - x_0)(x - x_1) \cdots (x - x_n)$

Then $p(x) = \varphi(x) \left(\dfrac{w_0 y_0}{x - x_0} + \dfrac{w_1 y_1}{x - x_1} + \dots + \dfrac{w_n y_n}{x - x_n} \right)$

or $= \varphi(x) \sum_{i=0}^{n} \dfrac{w_i}{x - x_i} y_i$ $(**)$

Here $w_i = \dfrac{1}{\prod_{j \neq i} x_i - x_j}$, $i = 0, 1, \dots, n$, are the

<u>barycentric weights.</u> in fact, same as $(*)$, but...

$(**)$ is the first barycentric interpolation formula. <u>Advantage:</u> once we compute w_i, we can evaluate p at any x w/ only $O(n)$ operations

But, formula (**) is not the end of the story:

If we have all y_i's equal to 1,

then $p(x) \equiv 1 \Rightarrow 1 = \varphi(x) = \sum_{i=0}^{n} \frac{w_i}{x - x_i} \quad \forall x$.

Solving for $\varphi(x)$:

$$\varphi(x) = \frac{1}{\sum_{i=0}^{n} \frac{w_i}{x - x_i}} \quad \text{and}$$

substituting into (**):

$$p(x) = \left(1 \Big/ \sum_{i=0}^{n} \frac{w_i}{x - x_i} \right) \left(\sum_{i=0}^{n} \frac{w_i}{x - x_i} y_i \right) \quad \text{or}$$

$$\boxed{p(x) = \left(\sum_{i=0}^{n} \frac{w_i}{x - x_i} y_i \right) \Big/ \left(\sum_{i=0}^{n} \frac{w_i}{x - x_i} \right)} \quad (***)$$

barycentric interpolation formula

(Note: the coefficients of y_i's are

$$\frac{w_i}{x - x_i} \Big/ \left(\sum_{i=0}^{n} \frac{w_i}{x - x_i} \right) \quad \text{and their sum is } 1!$$

That is, $p(x)$ is, in fact, a "weighted average" of y_i, the function values!

<u>Example</u> Same data pts as in the previous one: $(1, 2)$, $(2, 3)$, $(3, 6)$

First, compute <u>weights</u>:

$$w_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)} = \frac{1}{(1-2)(1-3)} = \frac{1}{2}$$

$$w_1 = \frac{1}{(x_1 - x_0)(x_1 - x_2)} = \frac{1}{(2-1)(2-3)} = -1$$

$$w_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} = \frac{1}{(3-1)(3-2)} = \frac{1}{2}$$

$$\Rightarrow \quad p(x) = \frac{\frac{w_0 y_0}{x-x_0} + \frac{w_1 y_1}{x-x_1} + \frac{w_2 y_2}{x-x_2}}{\frac{w_0}{x-x_0} + \frac{w_1}{x-x_1} + \frac{w_2}{x-x_2}}$$

$$= \left(\frac{\frac{1}{2}\cdot 2}{x-1} + \frac{(-1)\cdot 3}{x-2} + \frac{\frac{1}{2}\cdot 6}{x-3} \right) \Big/ \left(\frac{1}{2(x-1)} - \frac{1}{x-2} + \frac{1}{2(x-3)} \right)$$

= [to see it's the same $p(x)$ as previously multiply by $\varphi(x) = (x-1)(x-2)(x-3)$]

$$= \frac{(x-2)(x-3) - 3(x-1)(x-3) + 3(x-1)(x-2)}{\frac{(x-2)(x-3)}{2} - (x-1)(x-3) + \frac{1}{2}(x-1)(x-2)}$$

$$= \frac{x^2-5x+6 - 3(x^2-4x+3) + 3(x^2-3x+2)}{\frac{x^2-5x+6}{2} - (x^2-4x+3) + \frac{x^2-3x+2}{2}}$$

$$= \frac{x^2-2x+3}{1} = \boxed{x^2-2x+3}$$

Summary: Given (x_i, y_i), $i = 0, 1, \ldots, n$, find $y = p(x)$ s.t. $y_i = p(x_i)$ deg $p \le n$

① $\underbrace{p(x) = \sum_{i=0}^{n} y_i \left(\prod_{j\neq i} \frac{x-x_j}{x_i-x_j} \right)}_{\text{Lagrange form of } p}$ Takes $O(n^2)$ operations to evaluate p at a new x.

② $\underbrace{p(x) = \varphi(x) \sum_{i=0}^{n} \frac{w_i}{x-x_i} y_i}$ w/ $\varphi(x) = \prod_{j=0}^{n} (x-x_j)$ and weights $w_i = \underbrace{\frac{1}{\prod_{j\neq i}(x_i-x_j)}}$ takes $O(n^2)$ flops

modified Lagrange form (1st form of barycentric formula)

to find p at a new $x \to O(n)$ flops

③ $\underbrace{p(x) = \left(\sum_{i=0}^{n} \frac{w_i}{x-x_i} y_i \right) \Big/ \left(\sum_{i=0}^{n} \frac{w_i}{x-x_i} \right)}_{\text{barycentric interpolation formula}}$ (same w_i)

Same work as in ②