

Before finding a_0, a_1, \dots, a_n , let us (2)
try to estimate work needed to evaluate
 $p(x)$ at some x . Recall Horner's rule:

- Start from setting $y = a_n$
- For $i = n-1, n-2, \dots, 0$, $y \rightarrow y(x-x_i) + a_i$

(That is, $y = a_n$, then $y = a_n(x-x_{n-1}) + a_{n-1}$,
then $y = a_n(x-x_{n-1})(x-x_{n-2}) + a_{n-1}(x-x_{n-2}) + a_{n-2}$,
... , and at the end $y = p(x)$
 $= a_n \prod_{j=0}^{n-1} (x-x_j) + \dots + a_1(x-x_0) + a_0$.)

FLOPS at each step (out of n) is 3
(\oplus, \ominus, \otimes) \Rightarrow total number is $\underbrace{3n \text{ FLOPS}}_{O(n)}$
to evaluate $p(x)$.

Now, how do we find a_i 's?

We apply the conditions $y_i = p(x_i)$, $i = 0, \dots, n$,
one by one:

$$p(x_0) = \boxed{a_0 = y_0}$$

$$p(x_1) = a_0 + a_1(x_1 - x_0) = y_1 \Rightarrow \boxed{a_1 = \frac{y_1 - a_0}{x_1 - x_0}}$$

$$p(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = y_2$$
$$\Rightarrow \boxed{a_2 = \frac{y_2 - a_0 - a_1(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)}}, \dots \text{ etc.}!$$

We are solving the following system:

$$\begin{pmatrix}
 1 & 0 & 0 & \dots & 0 \\
 1 & x_1 - x_0 & 0 & \dots & 0 \\
 1 & x_2 - x_0 & (x_2 - x_0)(x_2 - x_1) & \dots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 1 & x_n - x_0 & (x_n - x_0)(x_n - x_1) & \dots & \prod_{j=0}^{n-1} (x_n - x_j)
 \end{pmatrix}
 \begin{pmatrix}
 a_0 \\
 a_1 \\
 a_2 \\
 \vdots \\
 a_n
 \end{pmatrix}
 =
 \begin{pmatrix}
 y_0 \\
 y_1 \\
 y_2 \\
 \vdots \\
 y_n
 \end{pmatrix}$$

lower triangular matrix

$$\begin{pmatrix}
 * & & & \\
 * & * & & \\
 * & * & * & \\
 * & * & * & *
 \end{pmatrix}$$

(It's more work to use the Vandermonde system!)

Work to solve this system is $O(n^2) + O(n^2)$ to compute the matrix

Advantage of the Newton form of $p(x)$:
 if we add another pt (x_{n+1}, y_{n+1}) , then the previously computed a_0, a_1, \dots, a_n do not change. We only need to find a_{n+1} . (and $p(x)$ has degree $n+1$ now). We do that by adding another row at the bottom and include a_{n+1} & y_{n+1} in the column vectors.

(Note: in the barycentric form of Lagrange polynomial adding (x_{n+1}, y_{n+1}) is easy enough: divide each w_i by $(x_i - x_{n+1})$)

and compute $w_{n+1} = \frac{1}{\prod_{j=0}^n (x_{n+1} - x_j)}$ (4)

Example: Consider pts $(1,2), (2,3), (3,6)$

(1) The Newton form is $p(x) = a_0 + a_1(x-1) + a_2(x-1)(x-2)$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2-1 & 0 \\ 1 & 3-1 & (3-1)(3-2) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{pmatrix}$$

$$\Rightarrow a_0 = 2 \\ a_0 + a_1 = 3 \Rightarrow a_1 = 1$$

$$a_0 + 2a_1 + 2a_2 = 6 \Rightarrow a_2 = 1$$

Thus, $p(x) = 2 + (x-1) + (x-1)(x-2)$

$$= x+1 + x^2 - x - 2x + 2 = \boxed{x^2 - 2x + 3} \quad \checkmark \quad \begin{matrix} \text{same!} \\ \text{as in §2.2} \end{matrix}$$

(2) One can choose a different order:

$$\begin{pmatrix} 3,6 \\ x_0 \end{pmatrix}, \begin{pmatrix} 1,2 \\ x_1 \end{pmatrix}, \begin{pmatrix} 2,3 \\ x_2 \end{pmatrix} \Rightarrow p(x) = b_0 + b_1(x-3) + b_2(x-3)(x-1)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1-3 & 0 \\ 1 & 2-3 & (2-3)(2-1) \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\Rightarrow b_0 = 6 \\ b_0 - 2b_1 = 2 \Rightarrow b_1 = 2$$

$$b_0 - b_1 - b_2 = 3 \Rightarrow b_2 = 1$$

So, $p(x) = 6 + 2(x-3) + (x-3)(x-1)$

$$= 6 + 2x - 6 + x^2 - 3x - x + 3 = \boxed{x^2 - 2x + 3} \quad \checkmark$$

(3) Adding a new point $(5, 7)$ to the nodes $(1, 2), (2, 3), (3, 6)$: find $q(x)$ of deg 3 (5)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2-1 & 0 & 0 \\ 1 & 3-1 & (3-1)(3-2) & 0 \\ 1 & 5-1 & (5-1)(5-2) & (5-1)(5-2)(5-3) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 6 \\ 7 \end{pmatrix}$$

see part (1) $\Rightarrow a_0 = 2, a_1 = 1, a_2 = 1$ are already found!

\Rightarrow to find a_3 : $a_0 + 4a_1 + 12a_2 + 24a_3 = 7 \Rightarrow a_3 = -\frac{11}{24}$

$\Rightarrow p(x) = 2 + (x-1) + (x-1)(x-2) - \frac{11}{24}(x-1)(x-2)(x-3)$.

§ 8.3.1 Divided Differences.

If some of nodes x_0, x_1, \dots, x_n are far apart, then the products $\prod_j (x_i - x_j)$ are likely to be huge and wight result in overflow.

If, on the contrary, the nodes are close together, the products $\prod_j (x_i - x_j)$ are small and can lead to underflow (result \ll smallest floating-pt. number).
 (result \gg largest floating-pt. number $(\approx 1.8 \times 10^{308})$)
 (result \ll smallest floating-pt. number)

Thus, another algorithm is often used in practice: (or a subnormal #)

Given (x_i, y_i) , $i=0, 1, \dots, n$, denote $y_i = f(x_i)$ (6)

or, simply, $y_i = f_i$.

Define $f[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$ to be called a k th order divided difference a coefficient of x^k in the polynomial of degree k that interpolates $(x_{i_0}, f_{i_0}), \dots, (x_{i_k}, f_{i_k})$. Newton form:

$$d_0 + d_1(x-x_{i_0}) + d_2(x-x_{i_0})(x-x_{i_1}) + \dots + d_k \prod_{j=0}^{k-1} (x-x_{i_j})$$

\downarrow $f[x_{i_0}]$ \rightarrow $f[x_{i_0}, x_{i_1}]$... $f[x_{i_0}, x_{i_1}, \dots, x_{i_k}]$

(Note: $f[x_0] = a_0$, $f[x_0, x_1] = a_1$, ..., $f[x_0, \dots, x_k] = a_k$ in §8.3.)

Taking $(x_i, y_i) = (x_i, f_i)$, $i=0, 1, \dots, n$, we can interpolate: (1) any single node x_i using a polynomial of deg 0: $p_{0,i}(x) = f_i$ as $p_{0,i}(x_i) = f_i$ ^{must} and, using our notation, $f_i = f[x_i] \rightarrow$ 0-order divided difference.

(2) any two nodes x_i and x_j ($i \neq j$) using a polynomial of deg. 1: 1st-order divided difference

$$p_{1,i,j}(x) = f[x_i] + f[x_i, x_j](x-x_i)$$

$$f_i = p_{1,i,j}(x_i)$$

$$\text{and } f[x_i, x_j] = \frac{f[x_j] - f[x_i]}{x_j - x_i}$$

(comes from Newton form §8.3)

③ Continuing in this fashion: for x_i, x_j, x_k ,

$$f[x_i, x_j, x_k] = \frac{f[x_j, x_k] - f[x_i, x_j]}{(2\text{nd order divided difference}) x_k - x_i}$$

and, more generally:

$$④ f[x_0, x_1, \dots, x_k] = \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

Why are we doing this?

Consider the table:

$f[x_0] = f_0$	$= a_0$		
$f[x_1] = f_1$	$f[x_0, x_1]$	$= a_1$	
$f[x_2] = f_2$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	$= a_2$
$f[x_3] = f_3$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$
			$= a_3$

The diagonal entries: coefficients in the Newton form! $p(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) +$

$$a_3(x-x_0)(x-x_1)(x-x_2)$$

The table is easy to use:

Take $(1, 2), (2, 3), (3, 6)$
 $x_0, f_0 \quad x_1, f_1 \quad x_2, f_2$

Then $f[x_0] = 2$

$$f[x_1] = 3 \quad f[x_0, x_1] = \frac{3-2}{2-1} = 1$$

$$f[x_2] = 6 \quad f[x_1, x_2] = \frac{6-3}{3-2} = 3 \quad f[x_0, x_1, x_2]$$

$$\Rightarrow p(x) = 2 + (x-1) + (x-1)(x-2) = x^2 - 2x + 3$$

$$\frac{3-1}{3-1} = 1$$

Another example:

Given data

i	x_i	f_i
0	-1	3
1	0	-4
2	1	5
3	2	-6

(7)

Construct the 3-rd degree polynomial that fits the data using divided differences.

$$f[x_0] = 3$$

$$f[x_1] = -4 \quad f[x_0, x_1] = \frac{-4-3}{0-(-1)} = -7$$

$$f[x_2] = 5 \quad f[x_1, x_2] = \frac{5-(-4)}{1-0} = 9 \quad f[x_0, x_1, x_2] = \frac{9-(-7)}{1-(-1)} = 8$$

$$f[x_3] = -6 \quad f[x_2, x_3] = \frac{-6-5}{2-1} = -11 \quad f[x_1, x_2, x_3] = \frac{-11-9}{2-0} = -10$$

$$\text{and } f[x_0, x_1, x_2, x_3] = \frac{-10-8}{2-(-1)} = -6$$

$$\text{So, } p_3(x) = 3 - 7(x+1) + 8(x+1)x - 6(x+1)x(x-1)$$

* Read proof of Thm. 8.3.1 (formula for the k th order divided difference).

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Pseudo-code:      Given  $(x_i, f(x_i)), i=0, 1, \dots, n$ 
{
  for  $i=0, 1, \dots, n$  do
     $d_i = f(x_i)$ 
  end
  for  $i=1, 2, \dots, n$  do
    for  $j=n, n-1, \dots, i$  do
       $d_j = (d_j - d_{j-1}) / (x_j - x_{j-1})$ 
    end
  end
end

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MATLAB: "polyfit" uses nodes x_0, \dots, x_n to form a Vandermonde system (§ 8.1) to find coefficients c_i in $p(x) = \sum_{i=0}^n c_i x^i$

↓ #4(a) in HW3

$$p = \text{polyfit}(x, y, n)$$

data ↑ degree

"polyval" evaluates the polynomial $p(x)$ at a given x : $y = \text{polyval}(p, x)$