§ 4.2, 4.3 Taylor's Theorem. Newton's method.

Recall: Taylor's Thm.

Consider $f \in C^n(X)$, and let $f^{(n+1)}$ exists on X. Then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},$$

where ξ is between x_0 & x.

That is, $$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x)}$$

$$R_n(x) = O\left((x-x_0)^{n+1}\right)$$

Big-O notation:

$$f(x) = O(g(x)) \text{ if } |f(x)| \le C\,|g(x)| \quad \forall x \in X^*$$

and some constant C. ("f grows at the order of $g(x)$")

In Taylor's Thm: $$\lim_{x \to x_0} \frac{R_n(x)}{(x-x_0)^{n+1}} = \frac{f^{(n+1)}(\xi)}{(n+1)!} = C$$

Note: for $f \in C^\infty(x)$, $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)(x-x_0)^k}{k!}$

(Example: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ w/ $x_0 = 0$)

Newton's Method (Newton-Raphson Method)

Idea: Newton (1669)
Recurrence relation: Joseph Raphson (1690)
Both iteration & derivatives: Thomas Simpson (1740)

So, this is another root-finding method.

From Taylor's Thm, $f(x) = \underbrace{f(x_0) + f'(x_0)(x-x_0)}_{P_1(x)} + R_1(x)$
(given x_0)

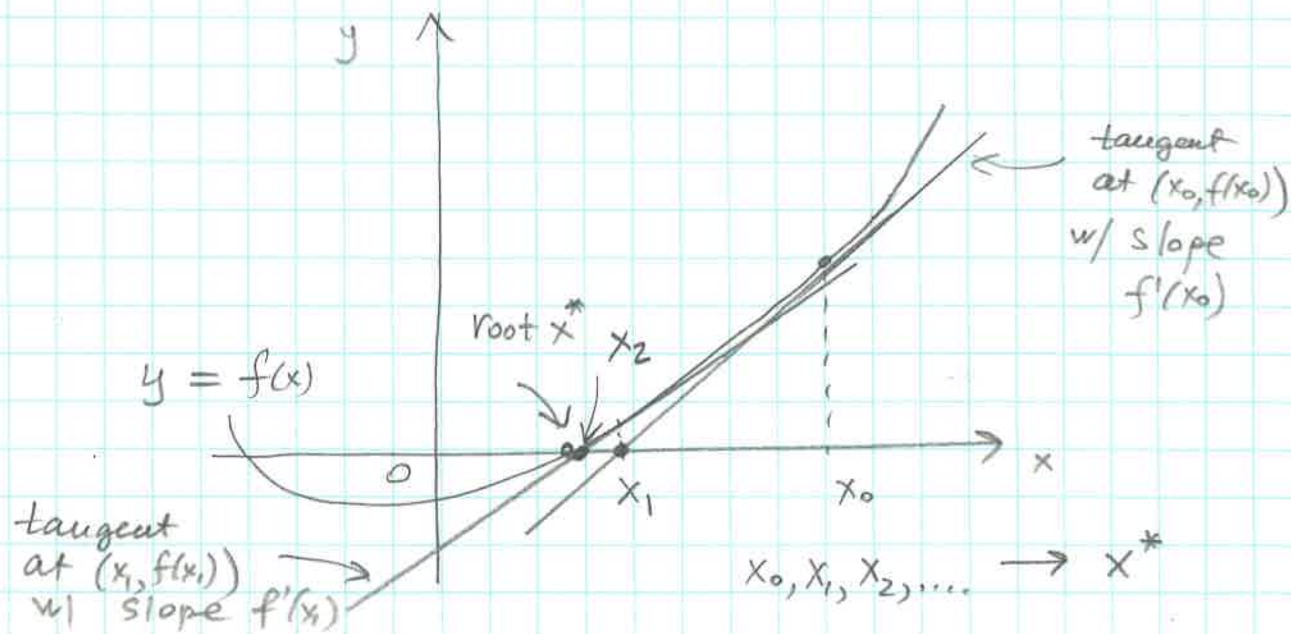
Note: $y = f(x_0) + f'(x_0)(x-x_0)$ is the tangent line to f at $(x_0, f(x_0))$. If $f'(x_0) \neq 0$, the line hits the x-axis at a pt. x_1:

$$0 = f(x_0) + f'(x_0)(x_1 - x_0) \Rightarrow$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$ Repeat! Find the tangent line at $(x_1, f(x_1)) \Rightarrow x_2, \ldots$

Thus, Newton's method is:

$$\boxed{\begin{array}{l} \text{Given } x_0, \text{ for } k = 0, 1, 2, \ldots, \\ \text{Set } \quad x_{k+1} = x_k - \dfrac{f(x_k)}{f'(x_k)} \end{array}}$$

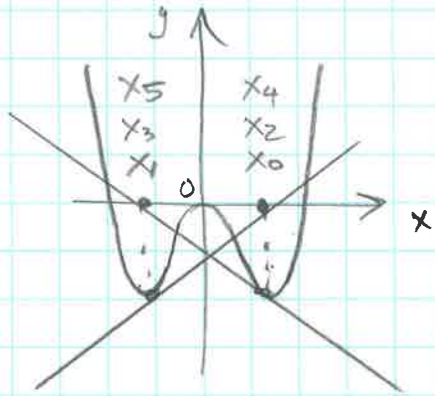


Newton's method does not always converge.

1. $f(x) = 4x^4 - 4x^2$ has roots $x=0$, $x=\pm 1$

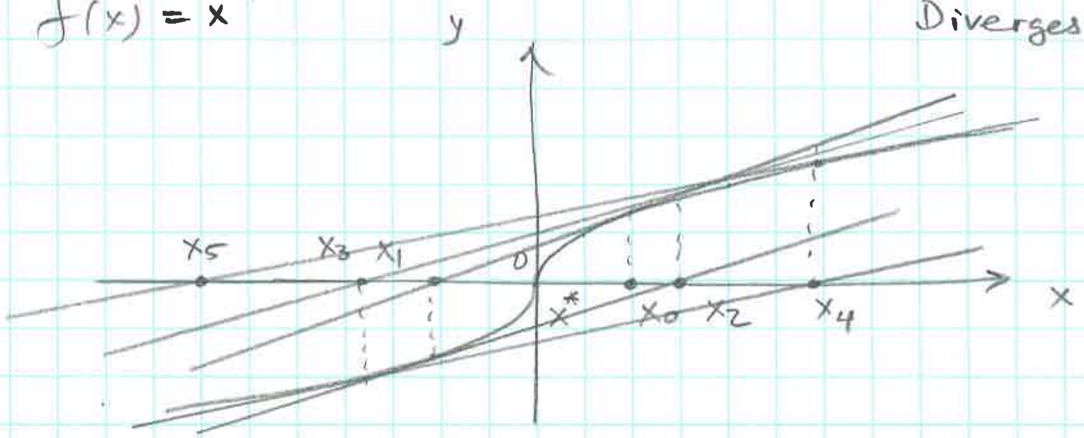
If $x_0 = \frac{\sqrt{21}}{7}$ then N.M. will alternate between $-\frac{\sqrt{21}}{7}$ & $\frac{\sqrt{21}}{7}$

$x_0 = x_2 = x_4 = \ldots$
$x_1 = x_3 = x_5 = \ldots$



2. $f(x) = x^{1/3}$

Diverges from x^*!



Combining N.M. w/ Bisection method:

Determine $[a,b]$ where $f(a)\cdot f(b) < 0 \Rightarrow$ solution is in $[a,b]$. For instance, if $x_0 \in (a,b)$, but $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \notin (a,b)$, apply bisection to find x_1!

(Can solve some bad cases, but not all of them.)

Back to Taylor's Thm: for a root x^*,

$$\underbrace{f(x^*)}_{0} = f(x_0) + f'(x_0)(x^*-x_0) + \underbrace{\frac{f''(\xi)}{2!}(x^*-x_0)^2}_{R_1(x^*)}$$

ξ between x_0 and x^*.

If x_0 is close to x^*, then $R_1(x^*)$ is small.

<u>Theorem (4.3.1)</u> Convergence of N.M.

If $f \in C^2$, and if x_0 is <u>sufficiently</u> close to a root x^* of f, and if $f'(x^*) \neq 0$, then N.M. converges to x^*, and ultimately the convergence is <u>quadratic</u>, i.e., there exists a constant $C^* = \left|\frac{f''(x^*)}{2f'(x^*)}\right|$ such that

$$\lim_{k\to\infty} \frac{|x_{k+1}-x^*|}{|x_k - x^*|^2} = C^*.$$

(That is, for k large enough, convergence is rapid! $\forall C > C^*$, there exists N s.t. $\forall k \geq N$, $|x_{k+1}-x^*| \leq C|x_k - x^*|^2$. Say, for $C=1$ & $|x_N - x^*| = 10^{-2}$, $|x_{N+1}-x^*| \leq 10^{-4}$!)

<u>Notes:</u>

1) "sufficiently close" $\rightarrow$ see proof

2) quadratic convergence does not explain how many steps are needed.

Proof:

$$0 = f(x^*) = f(x_0) + f'(x_0)(x^*-x_0) + \frac{f''(\xi)(x^*-x_0)^2}{2!}$$

Replace x_0 w/ x_k & solve for x^*:

$$(1) \quad x^* = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f''(\xi_k)}{f'(x_k)} \frac{(x^*-x_k)^2}{2}, \quad \text{w/ } \xi_k \text{ between } x_k \text{ \& } x^*$$

Also: $$x_{k+1} \overset{N.M.}{=} x_k - \frac{f(x_k)}{f'(x_k)} \quad (2)$$

Subtracting (1) from (2) gives:

$$x_{k+1} - x^* = \frac{f''(\xi_k)}{f'(x_k)} \frac{(x^*-x_k)^2}{2} \quad (3)$$

f'' is continuous, $f'(x^*) \neq 0$ (given) $\Rightarrow$

$\forall C > C^* = \left| \frac{f''(x^*)}{2f'(x^*)} \right|$, there is an interval I

about x^* in which $\left| \frac{f''(\xi)}{2f'(x)} \right| \leq C \ \forall x, \xi \in I$.

If, for some N, $x_N \in I$, and if also $|x_N - x^*| < 1/C$ (holds even for $N=0$, if x_0 is close to x^*), then from (3) we have:

$$|x_{N+1} - x^*| \leq C|x_N - x^*|^2 = C|x_N - x^*| \underbrace{|x_N - x^*|}_{< 1/C}$$

$< |x_N - x^*|$. Thus, $x_{N+1} \in I$ and also $|x_{N+1} - x^*| < 1/C$.

By induction, if all iterates x_k, $k \geq N$, lie in I, then $x_{k+1} \in I$ and

$$|x_{k+1} - x^*| \le C|x_k - x^*|^2 \le (C|x_k - x^*|)|x_k - x^*|$$
$$\le (C|x_k - x^*|)(C|x_{k-1} - x^*|^2) = (C|x_k - x^*|)(C|x_{k-1} - x^*|)|x_{k-1} - x^*|$$
$$\le C(|x_k - x^*|)(C|x_{k-1} - x^*|)(C|x_{k-2} - x^*|)|x_{k-2} - x^*|$$
$$\le \dots \le (\underbrace{C|x_N - x_*|}_{<1 \text{ (since } |x_N - x^*| < 1/C)})^{k+1-N}|x_N - x^*|$$

$$\Rightarrow C|x_N - x^*|^{k+1-N} \xrightarrow[k\to\infty]{} 0 \Rightarrow x_k \to x^* \text{ as } k \to \infty.$$

Also, since ξ_k is between x^* & x_k,

$$\xi_k \xrightarrow[k\to\infty]{} x^* \text{ and } \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \left|\frac{f''(\xi_k)}{2f'(x_k)}\right| \xrightarrow[k\to\infty]{}$$

$$\left|\frac{f''(x^*)}{2f'(x^*)}\right| = C^*$$

$\square$

$\Rightarrow$ "Sufficiently close": $|x_0 - x^*| < 1/C$ where $C \ge |f''/2f'|$

Some Examples:

1) $f(x) \equiv x^2 - 2 = 0$ has sol's $x = \pm\sqrt{2}$.

$$f'(x) = 2x \Rightarrow x_{k+1} = x_k - \frac{x_k^2 - 2}{2x_k}, \quad k = 0, 1, 2, \dots$$

$x_0 = 2$ and w/ error $e_k = |x_k - \sqrt{2}|$:

$x_0 = 2$	$e_0 = 0.59$
$x_1 = 1.5$	$e_1 = 0.086$
$x_2 = 1.4167$	$e_2 = 0.0025$
$x_3 = 1.4142157$	$e_3 = 2.1 \times 10^{-6} \approx 0.35 e_2^2$

$$C^* = \left|\frac{f''(\sqrt{2})}{2f'(\sqrt{2})}\right| = \frac{1}{2\sqrt{2}} \approx 0.3536$$

$$\left(\underbrace{|x_{k+1} - \sqrt{2}|}_{e_{k+1}} \le C^* \underbrace{|x_k - \sqrt{2}|}_{e_k}^2\right)$$

2) If $f'(x_*) = 0$, N.M. <u>may</u> converge, but only linearly: $f(x) \equiv x^2 = 0 \Rightarrow$

$f'(x) = 2x \Rightarrow$ N.M. is $x_{k+1} = x_k - \dfrac{x_k^2}{2x_k} = \dfrac{1}{2} x_k$

$x_0 = 1 \quad e_0 = 1$
$x_1 = e_1 = 1/2$
$x_2 = e_2 = \frac{1}{4}$
$\ldots \quad x_k = e_k = \frac{1}{2^k}$

(Error is reduced by a factor of 2)

3) $f_1(x) = \sin x$; $f_2(x) = \sin^2 x$; $x^* = \pi$ is a root!

N.M. for $\sin x = 0$:

$$x_{k+1} = x_k - \frac{\sin(x_k)}{\cos(x_k)}$$

$f_1'(x^*) = \cos\pi \neq 0$

$\Downarrow$

quadratic convergence

N.M. for $\sin^2 x = 0$:

$$x_{k+1} = x_k - \frac{\sin^2(x_k)}{2\sin(x_k)\cos(x_k)}$$

$$\left(= x_k - \frac{\sin(x_k)}{2\cos(x_k)} \right)$$

$f_2'(x^*) = 2\sin\pi\cos\pi = 0$

$(f_2''(x^*) \neq 0)$ $\Downarrow$

but: linear convergence

Why?

<u>Theorem (4.3.2)</u> If $f \in C^{p+1}$ for $p \geq 1$, and if x_0 is sufficiently close to a root x^*, and if $f'(x^*) = \cdots = f^{(p)}(x^*) = 0$, but $f^{(p+1)}(x^*) \neq 0$ then N.M. converges <u>linearly</u> to x^*, w/ the error reduced by about the factor $\frac{p}{p+1}$, that is,

$$\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|} = \frac{p}{p+1}.$$

(See proof in the text)