§ 10.2 Formula Based on Piecewise Polynomial Interpolation.

Divide $[a,b]$ into n subintervals $[\underset{\substack{\uparrow \\ a}}{x_0},x_1],\dots,[x_{n-1},\underset{\substack{\| \\ b}}{x_n}]$ and use the trapezoid rule to approximate each $\int_{x_{i-1}}^{x_i} f(x)\,dx$, then:

linear interpolant on ith subinterval

$$\int_a^b f(x)\,dx = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \overbrace{p_i(x)}\,dx$$

$$= \sum_{i=1}^n \Big[\underbrace{(x_i - x_{i-1}) \cdot \frac{f(x_i)+f(x_{i-1})}{2}}_{\text{trapezoid rule on } [x_{i-1},x_i]\text{, see § 10.1}} \Big]$$

If $x_i - x_{i-1} = h$ for all i (uniform mesh spacing), then

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n h\, \frac{f(x_i)+f(x_{i-1})}{2} = \underbrace{\frac{h}{2}\Big[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\Big]}_{\text{composite trapezoid rule}}$$

(here $f(x_i) = f_i$)

What about the error?

Recall: $\displaystyle\int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} p_i(x)\,dx = -\frac{1}{12}(\underbrace{x_i - x_{i-1}}_{=h})^3 f''(\xi_i)$, $\xi_i \in [x_{i-1},x_i]$

$= O(h^3)$ on $[x_{i-1},x_i]$

$$\Rightarrow \int_a^b f(x)\,dx - \sum_{i=1}^n \frac{h}{2}\Big[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\Big]$$

$$= \underset{\substack{\| \\ \frac{b-a}{h}}}{n} \cdot O(h^3) = O(h^2)$$

$\left(\text{as } h = \frac{b-a}{n}\right)$

Composite Simpson's Rule:

Again: $[a,b]$ is divided into $\underbrace{[x_0,x_1],\dots,[x_{n-1},x_n]}_{n \text{ subintervals}}$

then $$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \frac{x_i - x_{i-1}}{6}\Big[\underbrace{f(x_{i-1})}_{f_{i-1}} + 4\underbrace{f\Big(\frac{x_i+x_{i-1}}{2}\Big)}_{f_{i-1/2}} + \underbrace{f(x_i)}_{f_i}\Big]$$

$$= \frac{h}{6}\Big[(f_0 + 4f_{1/2} + f_1) + (f_1 + 4f_{2-1/2} + f_2) + \dots + (f_{n-1} + 4f_{n-1/2} + f_n)\Big]$$

if $x_i - x_{i-1} = h$ $\forall i$

$$= \frac{h}{6}\Big[f_0 + 4\underbrace{f_{1/2}}_{f\left(\frac{x_0+x_1}{2}\right)} + 2f_1 + 4\underbrace{f_{2-1/2}}_{f\left(\frac{x_1+x_2}{2}\right)} + 2f_2 + \dots + 2f_{n-1} + 4\underbrace{f_{n-1/2}}_{f\left(\frac{x_{n-1}+x_n}{2}\right)} + f_n\Big]$$

The error on each $[x_{i-1}, x_i]$ was shown to be

$$\frac{1}{2880}(x_i - x_{i-1})^5 f^{(4)}(x_{i-1}) + O\big((x_i - x_{i-1})^6\big) = O\big(\underbrace{(x_i - x_{i-1})}_{h}{}^5\big)$$

$= O(h^5)$ $\Rightarrow$ the total error is $n \times O(h^5)$

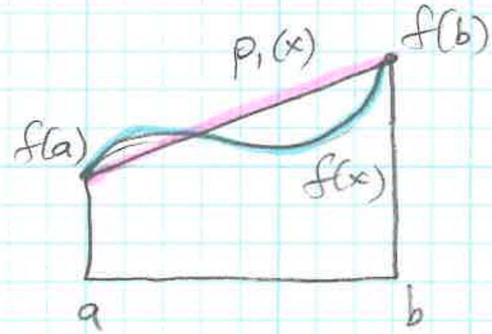
$$= \frac{b-a}{h} O(h^5) = O(h^4).$$

Summary: Quadrature formulas!

Method	Approximation of $\int_a^b f(x)\,dx$	Error
• Trapezoid Rule	$(b-a)\left(\frac{f(a)+f(b)}{2}\right)$	$-\frac{1}{12}(b-a)^3 f''(\xi)$
• Simpson's Rule	$\frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$	$\frac{1}{2880}(b-a)^5 f^{(4)}(\xi)$
• Composite trapezoid Rule	$\frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n\right]$	$O(h^2)$
• Composite Simpson's Rule	$\frac{h}{6}\left[f_0 + 4f_{1/2} + 2f_1 + \dots + 2f_{n-1} + 4f_{n-1/2} + f_n\right]$	$O(h^4)$

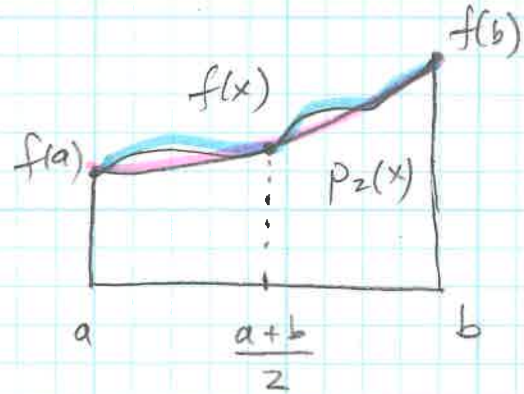
Quadrature Rules in pictures:

- Trapezoid rule:



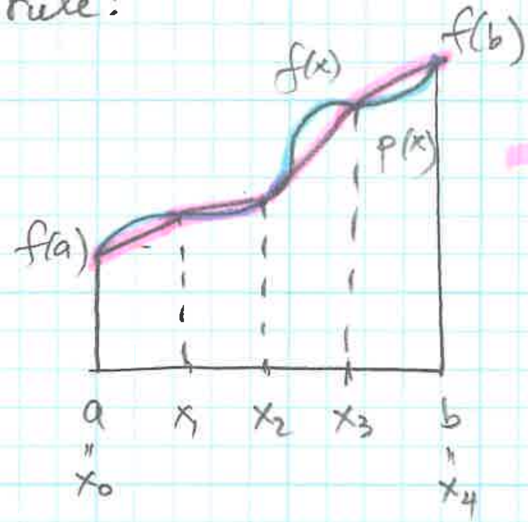
p_1 is linear

- Simpson's rule:



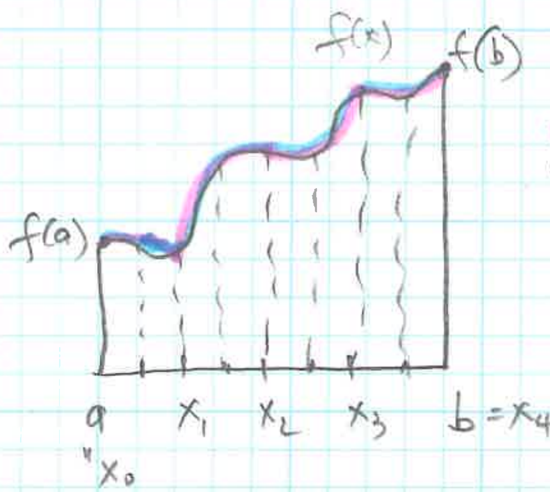
p_2 is quadratic

- Composite trapezoid rule:



p is piecewise linear

- Composite Simpson's rule:



p is piecewise quadratic