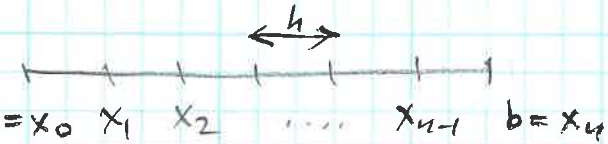
§ 8.6. Piecewise Linear Interpolation.

Let f be a function on $[a,b]$. We can approximate f by dividing $[a,b]$ into n subintervals, each of length $h = \frac{b-a}{n}$, and use a low-degree polynomial on each $[x_{i-1}, x_i]$, $i = 1, \dots, n$.



Consider $a = x_0, x_1, \dots, x_{n-1}, x_n = b$, then the linear interpolant of f on $[x_{i-1}, x_i]$ is

$$\underbrace{\ell(x)}_{\text{deg } 1} = \underbrace{f(x_{i-1}) \frac{x - x_i}{x_{i-1} - x_i} + f(x_i) \frac{x - x_{i-1}}{x_i - x_{i-1}}}_{\text{Lagrange form}}$$

Q: Why would we do this?

- One application is to evaluate $\int_a^b f(x)\,dx$ if it's hard (or even impossible) to evaluate analytically. Using polynomials is easy:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \Big[\underset{\overset{\shortparallel}{f(x_{i-1})}}{f_{i-1}} \frac{x - x_i}{x_{i-1} - x_i} + \underset{\overset{\shortparallel}{f(x_i)}}{f_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} \Big] dx$$

$$= \sum_{i=1}^{n} \left(f_{i-1} \cdot \frac{h}{2} + f_i \cdot \frac{h}{2} \right) = \underbrace{\frac{h}{2} \left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n \right]}_{\text{trapezoid rule (Ch. 10)}}$$

$$\Bigg(\int_{x_{i-1}}^{x_i} f_{i-1} \frac{x - x_i}{\underset{\overset{\shortparallel}{-h}}{x_{i-1} - x_i}}\,dx = -\frac{f_{i-1}}{h} \left(\frac{x^2}{2} - x_i \cdot x \right) \Big|_{x_{i-1}}^{x_i}$$

$$= -\frac{f_{i-1}}{h} \left[\frac{x_i^2}{2} - x_i^2 - \frac{x_{i-1}^2}{2} + x_i x_{i-1} \right] = \frac{f_{i-1}}{2h} \cdot h^2 = f_{i-1} \cdot \frac{h}{2} \Bigg)$$

– Another application:

Given $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, find $\sin(\pi/5)$:

Appoximate w/ $\ell(x) = \frac{1}{2}\,\frac{x-\pi/4}{\pi/6-\pi/4} + \frac{\sqrt{2}}{2}\,\frac{x-\pi/6}{\pi/4-\pi/6}$ $\Rightarrow$

$$\sin(\pi/5) \approx \ell(\pi/5) = \frac{1}{2}\cdot\frac{3}{5} + \frac{\sqrt{2}}{2}\cdot\frac{2}{5} \approx 0.58.$$

→ What about the error?

On $[x_{i-1}, x_i]$ $(i = 1, \dots, n)$,

$$|f(x) - \ell(x)| = \left|\frac{f''(\xi_x)}{2!}(x-x_{i-1})(x-x_i)\right| \le \quad (\xi_x \in [x_{i-1}, x_i])$$

$$\le \frac{M}{2}\,\underbrace{|(x-x_{i-1})(x-x_i)|}_{\text{this func. has max when } x = \frac{x_i+x_{i-1}}{2}} \le \frac{M}{2}\left(\frac{h}{2}\right)^2 = \frac{Mh^2}{8}$$

$\left(\text{if } |f''(x)| \le M \ \forall x \in [x_{i-1}, x_i]\right)$

So, if we want the error $\le \delta$, then from $\frac{Mh^2}{8} \le \delta \Rightarrow$ we have to choose $h < \sqrt{\frac{8\delta}{M}}$ for the length of subintervals.

→ What else can be done:

Use fewer subintervals, i.e. use fewer nodes in the regions where f is well-approx. by a line:

Starting w/ larger $[x_{i-1}, x_i]$, test if $\left|f\left(\frac{x_i+x_{i-1}}{2}\right) - \ell\left(\frac{x_i+x_{i-1}}{2}\right)\right| > \delta$, and if so, then divide $[x_{i-1}, x_i]$ into $\left[x_{i-1}, \frac{x_{i-1}+x_i}{2}\right], \left[\frac{x_{i-1}+x_i}{2}, x_i\right]$

midpoint.

→ What can get wrong? f may be close to ℓ at $\frac{x_i + x_{i-1}}{2}$, but far away otherwise.

But overall, this strategy works ok for many cases.

→ Another possibility: instead of linear $\ell(x)$, use a higher-degree interpolant on $[x_{i-1}, x_i]$, say $p(x)$ of deg 2. Since it's of degree 2, we need 3 nodes: x_{i-1}, $\frac{x_{i-1}+x_i}{2}$, x_i. Error?

$$f(x) - p(x) = \frac{f'''(\xi_x)}{3!} (x - x_{i-1})\left(x - \frac{x_{i-1}+x_i}{2}\right)(x - x_i)$$

w/ $\xi_x \in [x_{i-1}, x_i]$

If $|f'''(x)| \leq M$ then the error $= O(h^3)$.

<u>Note:</u> piecewise interpolants consisting of either linear ($\ell(x)$) or quadratic ($p(x)$) pieces are continuous, but their derivatives are not.

If we want a <u>smooth</u> interpolant $q(x)$, we ask $q(x_i) = f(x_i)$ and $q'(x_i) = f'(x_i)$.

If $\deg q = 2$, then:

$$\underbrace{q'(x)}_{\text{linear!}} = f'_{i-1} \frac{x - x_i}{x_{i-1} - x_i} + f'_i \frac{x - x_{i-1}}{x_i - x_{i-1}} \quad \text{on } [x_{i-1}, x_i]$$

$\Rightarrow$ $q(x) = \int_{x_{i-1}}^{x} q'(t)\,dt + C$ and to have $q(x_{i-1}) = f_{i-1}$

we need $C = f_{i-1}$, but if C is fixed, we have no way to force $q(x_i) = f_i$! $\Rightarrow$ an issue.

In the next sections: use <u>cubic</u> interpolants to get smooth approximations.