§ 9.2 Richardson Extrapolation

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Lewis Fry Richardson (1881 - 1953)

Another way to obtain higher-order accuracy in numerical differentiation. This method can be applied to many types of problems, e.g., approximating integrals, solving DEQ, etc.

Consider the centered-difference formula for f':

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

Saving more terms in the Taylor series for $f(x+h)$ and $f(x-h)$ gives: $(h>0)$

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + O(h^5)$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) + \frac{h^4}{4!} f^{(4)}(x) + O(h^5)$$

$\Rightarrow$ Subtracting and solving for $f'(x)$ results in

$$f'(x) = \underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{let } \varphi_0(h)} - \frac{h^2}{6} f'''(x) + O(h^4) \qquad \left(\frac{1}{2h} \cdot O(h^5)\right)$$

So, $f'(x) = \varphi_0(h) - \frac{h^2}{6} f'''(x) + O(h^4)$ (1)

Now let us use $h/2$ instead of h:

$$f'(x) = \varphi_0(h/2) - \frac{(h/2)^2}{6} f'''(x) + O(h^4) \qquad (2)$$

Multiplying the last eqn. by 4 and subtracting (2) yields: (we are getting rid of terms w/ h^2)

$$3 f'(x) = 4\varphi_0\left(\frac{h}{2}\right) - \varphi_0(h) + O(h^4)$$

$$\Rightarrow f'(x) = \frac{4}{3}\varphi_0\left(\frac{h}{2}\right) - \frac{1}{3}\varphi_0(h) + O(h^4)$$

$$\left(\overset{\text{or}}{=} \varphi_0\left(\frac{h}{2}\right) + \frac{\varphi_0(h/2) - \varphi_0(h)}{3} + O(h^4)\right)$$

Thus, $f'(x) \approx \underbrace{\frac{4}{3}\varphi_0(h/2) - \frac{1}{3}\varphi_0(h)}_{\text{difference formula } \varphi_1(h)}$ w/ trunc. err. $O(h^4)$

Requires twice as much work as $\underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\varphi_0(h)}$ of $O(h^2)$

but trunc. error decreases faster! $O(h^4)$.

Best accuracy $\underbrace{h^4}_{\text{trunc.}} + \underbrace{\frac{\varepsilon_m}{h}}_{\text{roundoff}}$ is achieved

when $h^4 \approx \frac{\varepsilon_m}{h}$

$\Rightarrow h \approx \sqrt[5]{\varepsilon_m} \Rightarrow$ the total error $O(h^4) \approx \varepsilon_m^{4/5}$.

This process can be repeated. If now

(3) $f'(x) = \varphi_1(h) + \underbrace{Ch^4 + O(h^5)}_{O(h^4)}$ is replaced w/

(4) $f'(x) = \varphi_1(h/2) + C(h/2)^4 + O(h^5)$ then

subtracting equation (3) from 16 times eqn. (4) gives:

$$15 f'(x) = 16\varphi_1\left(\frac{h}{2}\right) - \varphi_1(h) + O(h^5)$$

$$\Rightarrow f'(x) = \frac{16}{15}\varphi_1\left(\frac{h}{2}\right) - \frac{1}{15}\varphi_1(h) + \underbrace{O(h^5)}$$

in the error, $O(h^5)$ terms also cancel $\Rightarrow O(h^6)$

$$\Rightarrow f'(x) = \frac{16}{15}\varphi_1\left(\frac{h}{2}\right) - \frac{1}{15}\varphi_1(h) + O(h^6)$$

$$\left(\overset{\text{or}}{=} \varphi_1\left(\frac{h}{2}\right) + \frac{\varphi_1(h/2) - \varphi_1(h)}{15} + O(h^6)\right)$$

- Read Example 9.2.1, p. 223, w/ $f(x) = \sin x$.
- Another Example: $f(x) = 5xe^{-2x}$. Let $h = 0.25$, $x = 0.35$. Start from $f'(0.35) \approx \underbrace{\frac{f(0.35+0.25) - f(0.35-0.25)}{2 \cdot 0.25}}_{\text{centered difference formula}}$

recall: $O(h^2)$ $\downarrow$ $\varphi_0(0.25)$

$$\Rightarrow f'(0.35) \approx \frac{5(0.6)e^{-2(0.6)} - 5(0.1)e^{-2\cdot(0.1)}}{0.5}$$

$$= 0.9884 \quad (\text{using 4 places})$$

Now approx. $f'(0.35)$, using $h/2$, we find:

$$f'(0.35) \approx \underbrace{\frac{f(0.35+0.125) - f(0.35-0.125)}{2 \cdot 0.125}}_{\varphi_0(0.125)} \quad (\text{of } O(h^2))$$

$h = \frac{0.25}{2} = 0.125$

$$= \frac{5 \cdot (0.475) e^{-2\cdot(0.475)} - 5(0.225) \cdot e^{-2(0.225)}}{0.25} = 0.8047$$

So, $h = 0.25 \Rightarrow f'(0.35) \approx \varphi_0(0.25) = 0.9884$
$h = 0.125 \Rightarrow f'(0.35) \approx \varphi_0(0.125) = 0.8047$

Using Richardson's extrapolation (RE):

$$f'(0.35) \approx \frac{4}{3}\varphi_0(0.125) - \frac{1}{3}\varphi_0(0.25) \quad (O(h^4))$$

step 1

$$= \frac{4}{3} \cdot 0.8047 - \frac{1}{3} \cdot 0.9884 = 0.7435 = \varphi_1(0.25)$$

Thus, $h = 0.25 \Rightarrow f'(0.35) \approx 0.9884$
$h = 0.125 \Rightarrow f'(0.35) \approx 0.8047$

RE $\Rightarrow$ $f'(0.35) \approx \underbrace{0.7435}_{\text{higher-order accurate approximation}}$ $= \varphi_1(0.25)$ first-step of RE

Exact value $= \boxed{0.7449}$

Then, $f'(0.35) \approx \frac{16}{15}\varphi_1(0.125) - \frac{1}{15}\varphi_1(0.25) \quad (O(h^6))$

Step 2

$$= \frac{16}{15} \cdot \Big(\frac{4}{3}\underbrace{\varphi_0(0.0625)}_{0.7598} - \frac{1}{3}\underbrace{\varphi_0(0.125)}_{0.8047}\Big) - \frac{1}{15} \cdot 0.7427$$

$$= \boxed{0.7449}$$

Good!

Generally: If we want to approximate a value L w/ $\varphi_0(h)$ of some parameter h, and if $L = \varphi_0(h) + \underbrace{c_1 h + c_2 h^2 + \dots}_{O(h)}$, then

replacing h w/ $\frac{h}{2}$ gives:

$$L = \varphi_0(h/2) + \underbrace{\frac{c_1}{2} h + \frac{c_2}{4} h^2 + \dots}_{O(h)}$$

$$\Rightarrow L = \underbrace{2\varphi_0(h/2) - \varphi_0(h)}_{\text{let } \varphi_1(h)} \underbrace{- \frac{1}{2} c_2 h^2 - \frac{3}{4} c_3 h^3 + \dots}_{O(h^2)}$$

$$\Rightarrow L = \varphi_1(h) - \frac{c_2}{2} h^2 - \frac{3}{4} c_3 h^3 + \dots$$

Next, $L = \varphi_1(h/2) - \frac{c_2}{2}(h/2)^2 - \frac{3}{4} c_3 (h/2)^3 + \dots$

Multiplying it by 4 and subtracting the previous equation and dividing by 3 gives:

$$L = \underbrace{\frac{4}{3}\varphi_1\left(\frac{h}{2}\right) - \frac{1}{3}\varphi_1(h)}_{\varphi_2(h)} + \underbrace{\frac{1}{8} c_3 h^3 + \dots}_{O(h^3)}$$

Each step of RE increases k in $O(h^k)$ by one. If some powers of h are not present in the formula, then each RE step may increase the order of the error much faster.