# §10.5 Romberg Integration.

Recall Richardson extrapolation:

$$f(x) = \underbrace{\varphi_0(h)}_{\text{approx}} + \underbrace{O(h)}_{a_1 h + a_2 h^2 + \dots} \quad \text{(error)}, \quad \text{then} \quad h \to \frac{h}{2} \Rightarrow$$

$$f(x) = \varphi_0\left(\frac{h}{2}\right) + O(h) \Rightarrow \left(\text{eliminating } O(h) \text{ term}\right)$$

$$f(x) = \underbrace{\frac{4}{3}\varphi_0\left(\frac{h}{2}\right) - \frac{1}{3}\varphi_0(h)}_{\text{new approx. } \varphi_1(h)} + \underbrace{O(h^2)}_{\text{error}}$$

then repeat ... We will apply this idea to integration: start from the composite trapezoid rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f_0 + 2f_1 + 2f_2 + \dots + 2f_{n-1} + f_n\right] + \underbrace{Ch^2 + O(h^4)}_{O(h^2)}$$

[p. 233, §10.2] → no odd powers ($f \in C^\infty$)

Define: $T_h = \frac{h}{2}\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right]$ and now use the trapezoid rule w/ # intervals doubled, i.e. $h/2$:

$$\int_a^b f(x)\,dx = \overbrace{\frac{h}{4}\left[f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-1/2} + f_n\right]}^{T_{h/2}} + \frac{Ch^2}{4} + O(h^4)$$

(Notation: $f_{j-1/2} = f\left(\frac{x_{j-1}+x_j}{2}\right)$, e.g. $f_{1/2} = f\left(\frac{x_0+x_1}{2}\right)$, $f_{n-1/2} = f\left(\frac{x_{n-1}+x_n}{2}\right)$, etc.)

Now we eliminate $O(h^2)$ terms:

$$\int_a^b f(x)\,dx = \underbrace{\frac{4}{3} T_{h/2} - \frac{1}{3} T_h}_{\text{new approximation!}} + \underbrace{O(h^4)}_{\text{error}}$$ (2)

$$= \frac{h}{3}\left[f_0 + 2f_{1/2} + 2f_1 + \dots + 2f_{n-1} + f_n\right]$$
$$- \frac{h}{6}\left[f_0 + 2f_1 + \dots + 2f_{n-1} + f_n\right] + O(h^4)$$
$$= \underbrace{\frac{h}{6}\left[f_0 + 4f_{1/2} + 2f_1 + \dots + 2f_{n-1} + 4f_{n-1/2} + f_n\right]}_{\text{but... this is the composite Simpson's rule!}} + \overbrace{O(h^4)}^{\text{error}}$$

The process can be repeated to improve accuracy. In fact, $k$ in the error term $O(h^k)$ will be increased by 2 at a step.

So, Romberg integration is the trapezoid rule w/ repeated application of Richardson extrapolation.

See Ex. 10.5.1, p. 242 for $\int_0^2 e^{-x^2}dx$.

<u>Note:</u> MATLAB routine <u>romberg.m</u> is available, see p. 243 of text.