§7.4 Conditioning of Linear Systems

Recall: problem $f: X \to Y$ has condition numbers $\underset{(\text{abs.})}{C(x)} = |f'(x)|$ & $\underset{(\text{rel.})}{\kappa(x)} = \left|\frac{x f'(x)}{f(x)}\right|$

Problem of solving $Ax = b$:

input: A, b

output: x.

To measure the change in x due to changes in A & b, we need to introduce vector & matrix norms.

Def: A norm of a vector is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying, for all $v, w \in \mathbb{R}^n$,

(i) $\|v\| \geq 0$ with $\|v\| = 0 \iff v = 0$.

(ii) $\|\alpha v\| = |\alpha| \, \|v\|$ for any $\alpha \in \mathbb{R}$.

(iii) $\|v + w\| \leq \|v\| + \|w\|$ (triangle inequality)

Norms:

$$\|v\|_2 = \sqrt{\sum_{i=1}^{n} |v_i|^2} = \sqrt{\langle v, v \rangle},$$ the 2-norm or Euclidean norm

$$\|v\|_1 = \sum_{i=1}^{n} |v_i|,$$ the 1-norm.

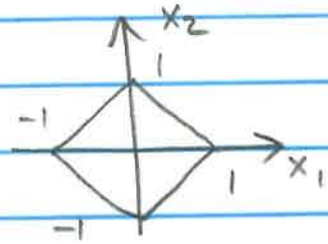
$$\|v\|_\infty = \max_{i=1,\dots,n} \{|v_i|\},$$ the ∞-norm

In general: $$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p},$$ the p-norm

The p-norms in $\mathbb{R}^2$ for $p = 1, 2,$ and ∞:
Consider the unit circle in p-norm:

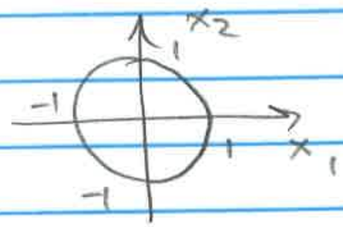
$\{x \in \mathbb{R}^2, \|x\|_p = 1\}$ Then:

$p = 1$



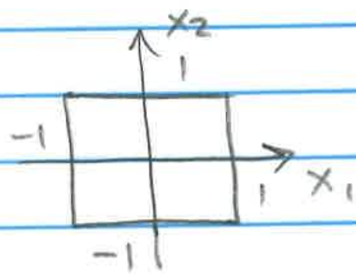
$|x_1| + |x_2| = 1$

$p = 2$



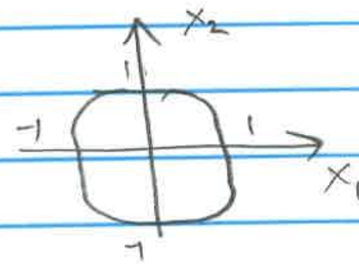
$x_1^2 + x_2^2 = 1$

$p = \infty$



$\max\{|x_1|, |x_2|\} = 1$

For some $1 < p < \infty$



Inequalities: for $v, w \in \mathbb{R}^n$,

$$\|v\|_\infty \le \|v\|_2 \le \sqrt{n}\,\|v\|_\infty$$
$$\|v\|_2 \le \|v\|_1 \le n\,\|v\|_\infty$$

Cauchy-Schwartz: $|v^T w| \le \|v\|_2 \|w\|_2$

Hölder inequality: $|v^T w| \le \|v\|_p \|w\|_q$, for $1/p + 1/q = 1$

Example: $v = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

$$\|v\|_2 = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$$
$$\|v\|_\infty = \max\{|1|, |2|, |-3|\} = 3$$
$$\|v\|_1 = |1| + |2| + |-3| = 6$$

MATLAB: $\|v\|_p$ is computed by norm (v, p)
($p = 2$ by default)

Matrix Norms

Def. A matrix norm is a function
$\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$, satisfying, for all $A, B \in \mathbb{R}^{m\times n}$,
(i) $\|A\| \geq 0$ with $\|A\| = 0 \iff A = 0$ (zero matrix)
(ii) $\|\alpha A\| = |\alpha| \, \|A\|$ for any $\alpha \in \mathbb{R}$
(iii) $\|A+B\| \leq \|A\| + \|B\|$ (triangle inequality)

Also, we assume (iv) $\|AC\| \leq \|A\| \|C\|$
($C \in \mathbb{R}^{n\times p}$), i.e., matrix norm is
submultiplicative.

- Given a vector norm $\|\cdot\|$, the induced matrix norm is $\|A\| = \max\limits_{\substack{\|v\|=1 \\ v\in\mathbb{R}^n}} \|Av\|$ or $= \max\limits_{\substack{v\neq 0 \\ v\in\mathbb{R}^n}} \dfrac{\|Av\|}{\|v\|}$
$A \in \mathbb{R}^{m\times n}$

Note: $\|A\| \geq \dfrac{\|Av\|}{\|v\|} \implies \|Av\| \leq \|A\| \|v\|$

Theorems 7.4.1 – 7.4.3 (pp. 155–157):

① The 1-norm of a matrix A is given by

$$\|A\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |a_{ij}| = \max_{j=1,\dots,n} \|\underbrace{A_j}_{j\text{-th column}}\|_1$$

called "maximum absolute column sum".

Let us show that $\|Av\|_1 \le \|A\|_1 \|v\|_1 \quad \forall v \in \mathbb{R}^n$

$$Av = v_1 A_1 + \dots + v_n A_n = \sum_{j=1}^{n} A_j v_j \quad \Rightarrow$$

$$\|Av\|_1 = \|v_1 A_1 + \dots + v_n A_n\| \le |v_1| \, \|A_1\|_1 + \dots + |v_n| \, \|A_n\|_1$$

$$\le \underbrace{(|v_1| + \dots + |v_n|)}_{\|v\|_1} \max_j \|A_j\|_1 = \underbrace{\max_j \|A_j\|_1}_{\|A\|_1} \|v\|_1,$$

That is $\|Av\|_1 \le \|A\|_1 \|v\|_1$.

Now, suppose we choose $v_0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \rightarrow j^*$ where j^* is s.t. $\max_j \|A_j\|_1 = \|A_{j^*}\|_1$

then $Av_0 = A_{j^*}$ & $\|v_0\| = 1$

Thus, $\|Av_0\|_1 = \|A_{j^*}\|_1 = \max_j \|A_j\|_1$

$= \max_j \|A_j\|_1 \underbrace{\|v_0\|_1}_{=1} = \|A\|_1 \|v_0\|_1$, then

we have a vector, $v_0 \neq 0$, for which the bound is attained.

2. The ∞-norm of a matrix A is given by

$$\underbrace{\|A\|_\infty}_{\text{called}} = \max_{i=1,\dots,m} \sum_{j=1}^{n} |a_{ij}| = \max_{i=1,\dots,m} \|\underbrace{a_i}_{\text{ith row}}\|_1$$

called "maximum absolute row sum".

(see p. 157)

3. The 2-norm of a matrix A is given by

$$\|A\|_2 = \sqrt{\underbrace{\lambda_{max}}_{}(A^TA)}$$

largest eig (A^TA)

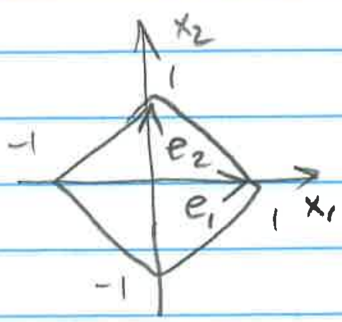
Note: $\|A\|_2^2 = \max_{\|v\|_2=1} \|Av\|_2^2 = \max_{\|v\|_2=1} (Av)^T(Av)$

$$= \max_{\|v\|_2=1} v^T(A^TA)v = \lambda_{max}(A^TA).$$

Geometric Insight: Consider $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$A : \mathbb{R}^2 \to \mathbb{R}^2$

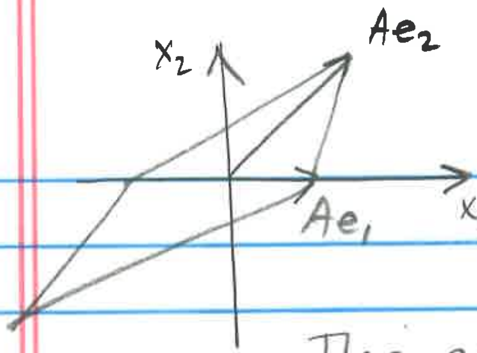
1. Unit ball in 1-norm: $\{x \in \mathbb{R}^2 \mid \|x\|_1 \le 1\}$



$$Ae_1 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e_1$$

$$Ae_2 = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Image of the unit ball under A:



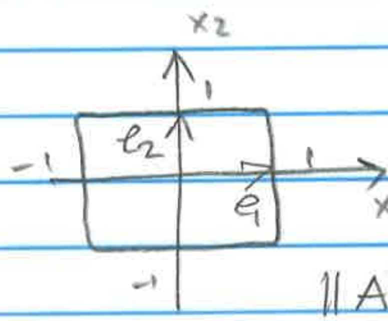
$\|Ax\|_1 \leq \|A\|_1 \|x\|_1$

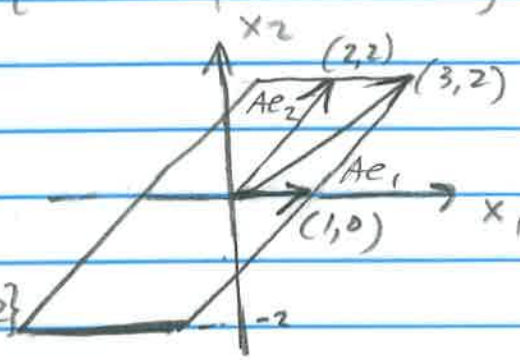
$\|A\|_1 = \max\{1+0, 2+2\} = 4$

So, $\|Ax\|_1 \leq 4\|x\|_1 \quad \forall x$

The equality is achieved when $x = e_2$ (or $-e_2$): $\|Ae_2\|_1 = 4$ and $4\|e_2\|_1 = 4$ (since $\|e_2\|_1 = 4$). That is, e_2 (and $-e_2$) is amplified most by A with factor 4.

② Unit ball in ∞-norm: $\{x \in \mathbb{R}^2 \mid \|x\|_\infty \leq 1\}$





$\|A\|_\infty = \max\{1+2, 0+2\} = 3$

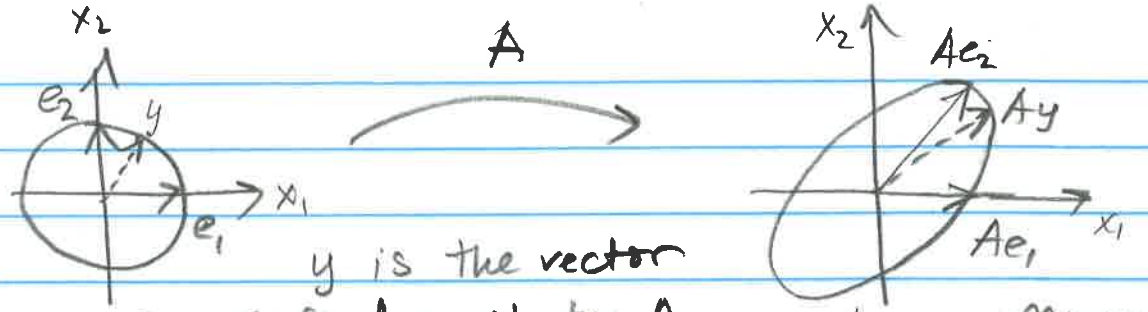
$\|Ae_1\|_\infty = 1, \quad \|Ae_2\|_\infty = 2, \quad \|A(1,1)^T\| = 3$

So, $\|Ax\|_\infty \leq 3\|x\|_\infty \quad \forall x$ and the equality is achieved for $x = \begin{pmatrix}1\\1\end{pmatrix} \rightsquigarrow$ vector amplified most by A with factor 3

$\left\|A\begin{pmatrix}1\\1\end{pmatrix}\right\|_\infty = \left\|\begin{pmatrix}3\\2\end{pmatrix}\right\|_\infty = 3, \quad \left\|\begin{pmatrix}1\\1\end{pmatrix}\right\|_\infty = 1, \quad \|A\|_\infty = 3$

③ Unit ball in 2-norm: $\{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^TA)} = \sqrt{\frac{9+\sqrt{65}}{2}} \approx 2.9208$$



y is the vector amplified most by A with factor 2.9208

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Read: Ex. 7.4.2, p. 157

MATLAB: $\|A\|_1 = \text{norm}(A,1)$

$\|A\|_2 = \text{norm}(A,2) = \text{norm}(A)$

$\|A\|_\infty = \text{norm}(A, \text{'Inf'})$

General matrix norm: does not have to be induced by a vector norm, but must satisfy the definition.

Another example: Frobenius norm

$A_{m \times n}$

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

$$= \left(\sum_{j=1}^{n} \|a_{ij}\|_2^2 \right)^{1/2}$$

$$= \sqrt{tr(A^T A)} = \sqrt{tr(AA^T)}$$