§7.5 Stability of Gaussian Elimination w/ Partial Pivoting.

$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa \varepsilon_m)$

Recall: A backward stable algorithm finds the exact solution to a nearby problem.

GE without partial pivoting may fail.

Consider $A = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix}$ (well-conditioned)

$\kappa_2(A) = \frac{3+\sqrt{5}}{2}$

$A = LU$ w/ $L = \begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix}$ & $U = \begin{pmatrix} 10^{-20} & 1 \\ 0 & 1-10^{20} \end{pmatrix}$

(no pivoting)

In floating-point arithmetic (w/ $\varepsilon_m \approx 10^{-16}$):

value $1-10^{20}$ will be rounded to the nearest fl.-pt. number, i.e. $fl(1-10^{20}) = -10^{20}$.

Then

$$\tilde{L} = fl(L) = \begin{pmatrix} 1 & 0 \\ 10^{20} & 1 \end{pmatrix} = L \text{, but}$$

$$\tilde{U} = fl(U) = \begin{pmatrix} 10^{-20} & 1 \\ 0 & -10^{20} \end{pmatrix} \neq U \quad \Rightarrow$$

$$\tilde{A} = \tilde{L}\tilde{U} = \begin{pmatrix} 10^{-20} & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 10^{-20} & 1 \\ 1 & 1 \end{pmatrix} = A \, !$$

If $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then sol. x to $Ax = b$ is $x \approx \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, while sol. $\tilde{x}$ to $\tilde{A}\tilde{x} = b$ is $\tilde{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- GE has computed LU stably, i.e. $\tilde{L}$ & $\tilde{U}$ are close to exact factors L & U of the matrix A. Yet, it did not solve $Ax=b$ stably, because GE without PP is not backward stable. The situation can be even worse, for instance, $\tilde{L}$ & $\tilde{U}$ can be ill-conditioned.

- Practically, GE with PP ($PA=LU$) is backward stable. There are exceptions, but they are extremely rare (see HW4, prob. 4).

- In general, how can one tell if an algorithm is backward stable?

$$Ax=b \longrightarrow \underbrace{(A+E)\hat{x}=\hat{b}}_{\text{nearby problem}}$$

Consider residual $b-A\hat{x}$. For a backward stable algorithm, $b-A\hat{x}$ should be small (even if the problem is ill-conditioned, i.e. $x-\hat{x}$ is not small)

$$b-A\hat{x} = b-A\hat{x}\pm\hat{b} = \underbrace{\hat{b}-A\hat{x}}_{E\hat{x}}+b-\hat{b} = E\hat{x}+b-\hat{b}$$

$$\Rightarrow \|b-A\hat{x}\| \le \|E\|\|\hat{x}\| + \|b-\hat{b}\| \quad \Rightarrow$$

$$\frac{\|b-A\hat{x}\|}{\|A\|\|\hat{x}\|} \le \frac{\|E\|}{\|A\|} + \frac{\|b-\hat{b}\|}{\|b\|}\cdot\frac{\|b\|}{\|A\|\|\hat{x}\|}$$

For $\|E\|$ & $\|b-\hat{b}\|$ small, the factor $\frac{\|b\|}{\|A\|\|\hat{x}\|}$ is less than or equal to ≈ 1

(this is because $\|b\| = \|Ax\| \leq \|A\|\|x\| \Rightarrow$

$\frac{\|b\|}{\|A\|\|x\|} \leq 1$), so $\frac{\|b-A\hat{x}\|}{\|A\|\|\hat{x}\|}$ is small

and independent of $\kappa(A)$ (!)

<u>Recall</u> Cholesky factorization: it is also backward stable ($A = LL^T$, no pivoting required).

<u>To summarize:</u>

Both conditioning & stability have effects on the accuracy of the solution of $Ax=b$.

No guarantee to achieve an accurate solution for ill-conditioned problems.