# MTH 331/431 Abstract Algebra I, Fall 2021
# FINAL EXAM
# due Monday, 12/20/21

| **YOUR NAME:** | **SCORE:** |
|---|---|

**Instructions:** This is a take-home exam. I expect your solutions to be well-written, neat, and organized. Do not turn in rough drafts. What you turn in should be the polished version of potentially several drafts. Feel free to type up your final version using LaTeX. The rules for the exam are:

1. **You should turn in this cover page** and all of the work that you have decided to submit. Please write your solutions on your own paper, show your work.
   **Start a new page for each problem.**
2. You may freely use any theorems and results that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 6.10 (Lagrange), then you should say so.
3. Unless you prove them, you cannot use any results that we have not covered.
4. You may use your textbook, notes, homework and midterm exam solutions, but you are NOT allowed to consult any other sources, when working on the exam. This includes your classmates, people outside of the class, and any online resources. *You are NOT allowed to copy someone else's work or let someone else copy your work. Someone else's solutions taken from the internet can often be easily detected - so, please DIY (do it yourself).*

*Attention:* You are given twelve problems, 10 points each. **MTH 331 students** are required to give solutions to **any eight problems**. **MTH 331&H and MTH 431 students** are required to give solutions to **any ten problems**.

To convince me that you have read and understand the instructions, sign in the box below:

**SIGNATURE:**

*Good luck and have fun!*

1. (10pts) Let $G = (0, \infty) \times \mathbb{R}$, and define a binary operation $*$ on $G$ by the formula $(a, b) * (c, d) = (ac, bc + d)$ for all $(a, b), (c, d) \in G$. Prove or disprove that $G$ forms a group under the operation $*$.

2. (10pts) Suppose $a$ and $b$ belong to a group, $a$ has odd order, and $aba^{-1} = b^{-1}$. Show that $b^2 = e$ (the identity of the group).

3. (10pts) Answer the following questions. Show your work.

   (a) What is the order of the factor group $(\mathbb{Z}_{12} \times U(12))/\langle(2, 7)\rangle$?

   (b) What is the order of the element $14 + \langle 8\rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8\rangle$?

4. (10pts) Let $G$ be a group, and let $H_1 \subseteq H_2 \subseteq \cdots$ be a nested sequence of subgroups of $G$. Prove that $\bigcup_{n=1}^{\infty} H_n$ is a subgroup of $G$.

5. (10pts) Let $G = \left\{ \begin{pmatrix} a & a-1 \\ 0 & 1 \end{pmatrix},\ a > 0 \right\}$.

   (a) Prove that $G$ is a subgroup of $GL_2(\mathbb{R})$.

   (b) Prove that $G$ is isomorphic to $\mathbb{R}$ (the group of real numbers under addition).

6. (10pts) Consider the group $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. Let $H = \langle(0, 2)\rangle$ and $K = \langle(1, 2)\rangle$.

   (a) Prove that $H$ is isomorphic to $K$.

   (b) Give the Cayley table of $G/H$.

   (c) Show that despite the fact that $H \cong K$, $G/H \ncong G/K$.

7. (10pts) Answer the following questions:

   (a) Is it possible that each element of an infinite group has a finite order? If so, give an example with an explanation. Otherwise, prove the non-existence of such a group.

   (b) Prove or disprove that an infinite group must have an infinite number of subgroups.

8. (10pts) Suppose $I$, $J$ are ideals of some ring $R$. Prove that $I \cap J$ and $I + J$ are both ideals of $R$.

9. (10pts) Let $\phi : R \to S$ be a ring homomorphism. Use induction to prove that for any $r \in R$ and any positive integer $n$, $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$.

10. (10pts) Prove that if $R$ is a field then every homomorphism from $R$ to a ring is either injective or maps everything onto 0.

11. (10pts) A ring $R$ is called a *Boolean ring* if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative.

12. (10pts) Note: a *subfield* $K$ of a field $F$ is a non-empty subset of $F$ that is a field under the operations of $F$. Let $F$ be a field and let $K$ be a subset of $F$ with at least two elements. Prove that $K$ is a subfield of $F$ if, for any $a$, $b$ ($b \neq 0$) in $K$, $a - b$ and $ab^{-1}$ belong to $K$.