MTH/CS 364/464 Numerical Analysis, Spring 2019
FINAL EXAM (Due at 10am on Thursday, 5/9/19)

Instructions: This is a take-home exam. You may *not* discuss the exam problems with anyone but me, the work should be yours only. Turn in this cover page and all of the work that you have decided to submit. Please write/print your solutions on your own paper, show your work. Start a new page for each problem and staple all the pages.

NAME: ______________________________	**SCORE:** __________

1. (12pts) Let $f(x) = e^{-x}$.
   (a) Find the n-th Taylor polynomial $P_n(x)$ about $x_0 = 0$.
   (b) Find the smallest value of n such that $|P_n(x) - f(x)| < 10^{-5}$ for all $x \in [0, 1]$.
   (c) Using n found in part (b), find the error of approximation of the value of $f(0.5)$ with $P_n(0.5)$.

2. (16pts) The equation $f(x) \equiv x^3 - 2x - 1 = 0$ has a root in $[1, 2]$.
   (a) Which of the functions below have a fixed point in the given interval? Show details.
   $$g_1(x) = \frac{2x+1}{x^2}, \quad g_2(x) = \sqrt{2 + 1/x}, \quad g_3(x) = x^3 - x - 1.$$
   (b) Starting with $x_0 = 1.5$, compute x_3 using the fixed point iteration for the appropriate function(s) from part (a).
   (c) Starting with $x_0 = 1.5$, compute x_3 using Newton's Method applied to $f(x)$.
   (d) Use Newton's Method to find the root, running until $|f(x_k)| \leq 10^{-6}$ (write your own MATLAB routine or use textbook routine newton.m). Count the number of iterations.

3. (12pts) Determine constants a, b, c, and d that will produce a quadrature formula
   $$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(1) + cf'(-1) + df'(1)$$
   that is exact for all polynomials of degree 3 or less.

4. (15pts) (Chapter 10, Exercise 6 of text) Consider the composite *midpoint rule* for approximating an integral
   $$\int_a^b f(x)dx \approx h\Sigma_{i=1}^n f\Big(\frac{x_i + x_{i-1}}{2}\Big),$$
   where $h = (b - a)/n$ and $x_i = a + ih$, $i = 0, 1, \ldots, n$.
   (a) Generate a plot in MATLAB to illustrate the midpoint rule geometrically, i.e., show what area is being computed by the formula (use any function f).

(b) Show this formula is exact if f is either constant or linear in each subinterval.

(c) Assuming that $f \in C^2[a,b]$, show that the midpoint rule is second-order accurate; that is, the error is less than or equal to a constant times h^2. To do this, you will first need to show that the error in each subinterval is of order h^3. To see this, expand f in a Taylor series about the midpoint $x_{i-1/2} = (x_i + x_{i-1})/2$ of the subinterval:

$$f(x) = f(x_{i-1/2}) + (x - x_{i-1/2})f'(x_{i-1/2}) + \frac{(x - x_{i-1/2})^2}{2} f''(\xi_{i-1/2}),\ \xi_{i-1/2} \in [x_{i-1}, x_i].$$

By integrating each term, show that the difference between the true value $\int_{x_{i-1}}^{x_i} f(x)\,dx$ and the approximation $hf(x_{i-1/2})$ is of order h^3. Finally, combine the results from all subintervals to show that the total error is of order h^2.

5. (15pts) Find the *natural* cubic spline S that interpolates the data $f(0) = 0$, $f(1) = 2$, $f(2) = 1$, $f(3) = 0$. Graph the spline with the data points (generate your graph in MATLAB).

6. (15pts) (Chapter 10, Exercise 8 of text) Continuing the love saga from Section 11.2.6, Juliet's emotional swings lead to many sleepless nights, which consequently dampens her emotions. Mathematically, the pair's love can now be expressed as

$$\begin{aligned} dx/dt &= -0.2y, \\ dy/dt &= 0.8x - 0.1y, \end{aligned}$$

Suppose this state of the romance begins when Romeo is smitten with Juliet ($x(0) = 2$) and Juliet is indifferent ($y(0) = 0$).

(a) Explain how these equations reflect Juliet's dampened emotions (compare to equations (11.23) in the textbook).

(b) Use *ode45* to produce three graphs in MATLAB, like those in Figure 11.7 in the textbook, showing Romeo and Juliet's love for $0 \le t \le 60$.

(c) From your graphs, describe how the change in Juliet described in this exercise will affect the relationship and its eventual outcome?

7. (15pts) Consider the initial value problem

$$y' = \frac{-ty}{\sqrt{2 - y^2}},\ \ 0 \le t \le 5,\ y(0) = 1.$$

(a) Write your own MATLAB routine to approximate the solution using the classical fourth-order Runge-Kutta method with $h = 0.25$. Graph the approximations.

(b) Solve the IVP using *ode45*. Graph the solution found.

HONOR PLEDGE:

I affirm that I have not given, received, or used any unauthorized assistance on this exam.

Your Signature: